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# SEVEN–SPHERE AND THE EXCEPTIONAL N=7 AND N=8 SUPERCONFORMAL ALGEBRAS

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## Abstract

We study realizations of the exceptional non-linear (quadratically generated, or  $W$ -type)  $N = 8$  and  $N = 7$  superconformal algebras with  $Spin(7)$  and  $G_2$  affine symmetry currents, respectively. Both the  $N = 8$  and  $N = 7$  algebras admit unitary highest-weight representations in terms of a single boson and free fermions in  $\underline{8}$  of  $Spin(7)$  and  $\underline{7}$  of  $G_2$ , with the central charges  $c_8 = 26/5$  and  $c_7 = 5$ , respectively. Furthermore, we show that the general coset Ansätze for the  $N = 8$  and  $N = 7$  algebras naturally lead to the coset spaces  $SO(8) \times U(1)/SO(7)$  and  $SO(7) \times U(1)/G_2$ , respectively, as the additional consistent solutions for certain values of the central charge. The coset space  $SO(8)/SO(7)$  is the seven-sphere  $S^7$ , whereas the space  $SO(7)/G_2$  represents the seven-sphere with torsion,  $S^7_T$ . The division algebra of octonions and the associated triality properties of  $SO(8)$  play an essential role in all these realizations. We also comment on some possible applications of our results to string theory.

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# 1 Introduction

Infinite conformal symmetry in two dimensions is the fundamental underlying symmetry of string theory [1], and it plays an essential role in the understanding of the critical behaviour of two-dimensional physical systems. Similarly, supersymmetric extensions of the infinite-dimensional conformal algebra underlie various superstring theories. For example, space-time supersymmetric classical vacua of superstring theories in various dimensions are described by extended superconformal field theories. Extended superconformal symmetry also has applications to integrable systems and to topological field theories as well [2, 3].

The finite-dimensional (global) subgroup of the two-dimensional conformal group  $SO(2, 2)$  is not simple, and it decomposes as  $SO(2, 2) \simeq SO(2, 1) \times SO(2, 1)$ , with the two  $SO(2, 1)$  factors acting on left and right movers, respectively. This allows one to have different number of supersymmetries in the left and right moving sectors. A complete classification of supersymmetric extensions of the *finite*-dimensional global conformal group in two dimensions was given in [4]. However, not all finite-dimensional (global) superconformal algebras admit extensions to *infinite*-dimensional linear superconformal algebras with generators of non-negative conformal dimensions. The maximal number ( $N$ ) of supersymmetries such linear infinite superconformal algebras can have is four [5, 6, 7, 8].

It is possible to have supersymmetric extensions of the Virasoro algebra with  $N > 4$ , while retaining the requirement that the generators have non-negative conformal dimensions. However, the price one has to pay is either to have a non-linear superconformal algebra, as in the Bershadsky-Knizhnik algebras [9, 10], or introduce field-dependent structure constants, as in the so-called *soft*  $N = 8$  algebra introduced in ref. [11] and

further studied in refs. [12, 13, 14, 15]. The soft  $N = 8$  algebra appears as the algebra of first-class constraints in the Green-Schwarz superstring action in ten dimensions [12]. Such field-dependent ‘structure constants’ also appear in the symmetry algebras of the two-dimensional *locally* supersymmetric non-linear sigma-models (NLSMs) with  $N > 4$ , where they may even become non-chiral due to non-trivial mixing between the left- and right-moving modes via dilaton couplings, when the number of supersymmetries exceeds eight [16]. The Grassmannian symmetric spaces

$$\frac{SO(8, m)}{SO(8) \times SO(m)} \quad m \geq 1, \quad (1.1)$$

appear as solutions for the  $N = 8$  locally supersymmetric NLSM target manifolds [17, 18]. The soft algebras usually have no restrictions on their central extensions, while their ‘structure constants’ are, in fact, functions on the target manifold.

The  $N$ -extended superconformal algebras of the type introduced by Bershadsky and Knizhnik [9, 10] comprise generators of conformal dimension 2,  $3/2$  and 1 only. They contain (i) the Virasoro subalgebra, (ii)  $N$  real supercurrents of conformal dimension  $3/2$ , whose operator products give the stress tensor of dimension 2, symmetry currents of dimension 1, and terms that are quadratic in the symmetry currents, (iii) satisfy the Jacobi ‘identities’, and (iv) have the usual spin-statistics relation. Under the requirement of reductivity, a complete classification of such algebras was given in refs. [19, 20]. Being “reductive” means that they linearise in the limit when their central charges go to infinity. In this limit, the infinite-dimensional vacuum-preserving algebra becomes a finite superalgebra containing the finite (global) conformal algebra. Thus, the full classification of such non-linear superconformal algebras [19, 20] follows from the classification of finite-dimensional (global) superconformal algebras given in ref. [4]. There are three

infinite classical families (for either the right- or the left-moving modes) ,

$$osp(N|2; \mathbf{R}) , \quad su(1, 1|N) , \quad osp(4^*|2N) , \quad (1.2)$$

a one-parameter family of the  $N = 4$  algebras, and two exceptional superconformal algebras with  $N = 7$  and  $N = 8$  supersymmetries.

All the extended superconformal algebras can be viewed as arising from the quantum hamiltonian (Drinfeld-Sokolov-type) reduction of affine Lie superalgebras [21]. In particular, the vacuum-preserving subalgebras of the  $N = 7$  and  $N = 8$  exceptional superconformal algebras in the limit of infinite central charge are the exceptional finite Lie superalgebras  $G(3)$  and  $F(4)$  (in the Kač notation [22]).<sup>3</sup> Hence, it is not surprising that the quantum hamiltonian reduction of the affine Lie superalgebras  $\widehat{G(3)}$  and  $\widehat{F(4)}$  just yields the  $N=7$  and  $N=8$  superconformal algebras, respectively [21]. The orthogonal and unitary series of eq. (1.2) are often referred to as the Bershadsky-Knizhnik superconformal algebras [9, 10]. The non-linear  $N = 4$  superconformal algebras were first obtained from the linear  $N = 4$  superconformal algebra by factoring out four free fermions and one boson [27, 28]. The infinite classical family of non-linear superconformal algebras corresponding to  $su(1, 1|N)$  for  $N > 2$  does not admit unitary representations of the highest-weight type [9, 29]. Similarly, the non-linear superconformal algebras corresponding to the symplectic series  $osp(4^*|2N)$  do not admit unitary representations for  $N > 1$  either [20]. The BRST operators of the Bershadsky-Knizhnik-type superconformal algebras were studied in refs. [30, 31, 32].

The main purpose of our investigation in this paper is to study possible coset space realizations for the  $N = 7$  and  $N = 8$  exceptional superconformal algebras. Thereby

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<sup>3</sup>See refs. [23, 24, 25] for details about the  $G(3)$  and  $F(4)$ . Another real form of  $F(4)$  corresponds to  $N = 2$  superconformal symmetry in five space-time dimensions [26].

we will also answer the question as to whether or not there exist rational (unitary) superconformal field theories with the ‘exceptional’  $N = 7$  or  $N = 8$  supersymmetry. When using the method of quantum hamiltonian reduction, the standard Wakimoto construction known for any affine Lie (super)algebra [2, 3] allows one to obtain a free field (Feigin-Fuchs) representation for any extended non-linear superconformal algebra [21]. Though being practical for a calculation of the screening operators as well as the correlation functions in superconformal field theory, using this method for constructing unitary highest-weight irreducible representations requires some additional techniques. For example, one still needs to find zeroes of the Kač determinant associated with a given (Verma) module and its null (singular) vectors, which may be a hard problem for the extended superconformal algebras at large  $N$ . On the other hand, the coset construction can, in principle, answer the question of existence of rational superconformal field theories in a relatively simple and straightforward way. Therefore, we shall assume that all the conformal fields in our construction come from the gauged  $(1,0)$  supersymmetric *Wess-Zumino-Novikov-Witten* (WZNW) models.

Many of the results about unitary highest-weight representations of the *linear*  $N = 2$  and  $N = 4$  superconformal algebras were obtained in the past by using known results concerning superconformal algebras with lower  $N$ . In the non-linear case, this method is obviously of a limited use, since the naive tensoring of representations is no longer valid. Thus we shall use the coset space method directly, in studying the unitary highest-weight representations of the exceptional non-linear superconformal algebras. The coset construction is well-known to be a powerful tool in the two-dimensional (super)conformal field theory [33], and it is presumably able to deliver all rational theories (modulo a permutation of fusion rules, or making an orbifold from the coset by modding it with

respect to a discrete not-free-acting symmetry [2, 3]). The generalizations of the coset space method are known for the  $N = 2$  extended supersymmetry [34], as well as for the  $N = 4$  supersymmetry [35, 36, 37, 38]. Though the coset space methods were invented to study representations of the linear extended superconformal algebras, they have also been extended to the non-linear  $N = 4$  superconformal algebras as well [35, 36, 37, 38].

Our paper is organized as follows. In sect. 2, we provide the necessary algebraical and group-theoretical background in seven and eight dimensions, which simultaneously introduces our notation. In sect. 3, we review the non-linear  $N = 7$  and  $N = 8$  superconformal algebras, using the language of the *operator product expansions* (OPEs). Our main results about the  $N = 8$  and  $N = 7$  coset constructions are presented in sect. 4. Our conclusions are summarized in sect. 5. The two appendices provide relevant identities for the octonionic structure constants (Appendix A), and some details about the supersymmetry part of the  $N = 8$  exceptional superconformal algebra (Appendix B).

## 2 A review of the properties of octonions, their automorphism group and gamma matrices in seven dimensions

In this section we shall review some known results about the division algebra of octonions, its automorphism group  $G_2$ , and gamma matrices in seven dimensions. These results will be used in later sections, in our study of the exceptional superconformal algebras.

### 2.1 Division algebra of octonions

Many of the special properties of various mathematical structures in seven and eight dimensions are related to the octonions. The eight-dimensional division algebra of oc-

tonions  $\mathbf{O}$  is one of the four division algebras that exist over the real numbers. An arbitrary octonion  $q$  can be expanded as [39]

$$q = \sum_{a=0}^7 q_a \hat{e}_a , \quad \text{all } q_a \text{ are real numbers} , \quad (2.1)$$

where  $\hat{e}_0 = 1$  represents the identity element, and the imaginary octonion units  $\hat{e}_m$ ,  $m = 1, 2, \dots, 7$ , satisfy the multiplication rule

$$\hat{e}_m \hat{e}_n = -\delta_{mn} + C_{mnp} \hat{e}_p . \quad (2.2)$$

with  $C_{mnp}$  being the totally antisymmetric structure constants. The seven imaginary units close under commutation. However, they do not form a Lie algebra under commutation due to the non-associativity of octonions. The ‘Jacobian’ of three elements is given by <sup>4</sup>

$$[\hat{e}_m, [\hat{e}_n, \hat{e}_p]] + \text{cyclic permutations} = 3C_{mnpq} \hat{e}_q , \quad (2.3)$$

where the non-vanishing totally antisymmetric tensor  $C_{mnpq}$  is defined as

$$C^k_{[mn} C_{p]kq} = C_{mnpq} \neq 0 . \quad (2.4)$$

The tensor  $C_{mnpq}$  is dual to the tensor  $C_{mnp}$  in seven dimensions:

$$C_{mnpq} = \frac{1}{6} \varepsilon_{mnpqrst} C_{rst} . \quad (2.5)$$

We use the basis given in ref. [39], for which the constants  $C_{mnp}$  read as:

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1 , \quad (2.6)$$

while all the other non-vanishing components are determined by the total antisymmetry.

Given eq. (2.6), the non-vanishing (equal to one) components of  $C_{mnpq}$  are given by the

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<sup>4</sup>We do not distinguish between co- and contra-variant indices. All (anti)symmetrizations are defined with unit weight.

following values of  $(mnpq)$  [24]:

$$\begin{aligned} (1, 2, 7, 6) , \quad (1, 2, 4, 5) , \quad (1, 3, 4, 6) , \quad (1, 3, 5, 7) , \\ (2, 3, 7, 4) , \quad (2, 3, 5, 6) , \quad \text{and} \quad (4, 5, 6, 7) , \end{aligned} \quad (2.7)$$

with the rest being fixed by the total antisymmetry.

The identities satisfied by the tensors  $C^{mnp}$  and  $C^{mnpq}$  have been extensively studied in the literature [39, 24, 40, 41, 42, 43]). Among them, one has

$$\{C^p, C^q\}_{mn} \equiv C^p_{mk} C^q_{kn} + C^q_{mk} C^p_{kn} = \delta_m^p \delta_n^q + \delta_n^p \delta_m^q - 2\delta^{pq} \delta_{mn} , \quad (2.8)$$

and

$$C_{mnpq} = -C^k_{mn} C^k_{pq} - \delta_{mq} \delta_{np} + \delta_{mp} \delta_{nq} . \quad (2.9)$$

A list of other useful identities, extending those found in refs. [39, 40, 41, 42, 43], is given in Appendix A.

## 2.2 $G_2$ , $Spin(7)$ , $SO(8)$ , and Octonions

The automorphism group of octonions is the exceptional group  $G_2$ . The automorphisms together with left and right multiplications by unit octonions generate the group  $SO(8)$  [39]. The operation of simultaneous multiplication from the left by a unit octonion  $q$  and right multiplication by the octonion conjugate  $\bar{q}$  together with the automorphisms generate the group  $SO(7)$  [39].

The  $8 \times 8$  gamma matrices  $\gamma_{ab}^i$  in seven dimensions, which satisfy the Clifford algebra:

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij} \mathbf{1}_8 , \quad (2.10)$$

where  $i, j, \dots = 1, 2, \dots, 7$ , and  $a, b, \dots = 1, 2, \dots, 8$ , can be written in terms of the octonionic structure constants [39, 40, 41, 42]. First, let's trivially extend  $C_{jk}^i$  to  $C_{ab}^i$



by setting  $C_{ab}^i = C_{jk}^i$  whenever  $a(=j)$  and  $b(=k)$  are not equal to 8, while defining  $C_{ab}^i$  to be zero whenever  $a$  or  $b$  is equal to 8. The hermitian (purely imaginary and antisymmetric) gamma matrices in seven dimensions can then be chosen as

$$\gamma_{ab}^i = i \left( C_{ab}^i \pm \delta_{ia} \delta_{b8} \mp \delta_{ib} \delta_{a8} \right) , \quad (2.11)$$

where the signs are correlated. Both options for the signs in eq. (2.11) will be exploited in sect. 4. In later sections we shall use the notation  $\gamma_{ab}^i$  for the upper sign choice, whereas the notation  $\tilde{\gamma}_{ab}^i$  is going to be used for the lower sign choice, in order to avoid confusion.

The antisymmetric products of gamma matrices are defined as usual, with unit weight, *viz.*

$$\gamma^{ij\dots k} = \gamma^{[i} \gamma^j \dots \gamma^{k]} . \quad (2.12)$$

The antisymmetric self-dual and antiself-dual tensors  $C_{IJKL}^\pm$ , ( $I, J, \dots = 1, 2, \dots, 8$ ) in eight dimensions will be defined as in refs. [40, 41, 42]:

$$C_{ijkl}^\pm = C_{ijkl} , \quad \text{and} \quad C_{ijk8}^\pm = \pm C_{ijk} , \quad (2.13)$$

With the above choices of gamma matrices one finds

$$\begin{aligned} \gamma_{ab}^{ij} &= C_{ijab} + \delta_a^i \delta_b^j - \delta_b^i \delta_a^j \pm C_a^{ij} \delta_{b8} \mp C_b^{ij} \delta_{a8} \\ &= C_{ijab}^\pm + \delta_a^i \delta_b^j - \delta_b^i \delta_a^j . \end{aligned} \quad (2.14)$$

A bit more effort is needed to calculate  $\gamma^{ijkl}$ , and we summarize some of the details below. First, it is straightforward to verify that

$$\begin{aligned} \gamma_{ac}^{ij} \gamma_{cb}^{kl} &= -C_{ijac}^\pm C_{klbc}^\pm + C_{ijak}^\pm \delta_{lb} - C_{ijal}^\pm \delta_{kb} + C_{kljb}^\pm \delta_{ia} \\ &\quad - C_{klib}^\pm \delta_{ja} + \delta_{ia} \delta_{jk} \delta_{lb} - \delta_{ia} \delta_{jl} \delta_{kb} - \delta_{ik} \delta_{ja} \delta_{lb} + \delta_{il} \delta_{ja} \delta_{kb} , \end{aligned}$$

and, hence,

$$\begin{aligned}\gamma_{ac}^{[ij} \gamma_{cb}^{kl]} &= -C_{ac}^{\pm [ij} C_{bc}^{\pm kl]} - 2C_{[ijk}^{\pm a} \delta_l^b] - 2C_{[ijk}^{\pm b} \delta_l^a] \\ &= -C_{ap}^{\pm [ij} C_{bp}^{\pm kl]} - C_a^{[ij} C_b^{kl]} - 2C_{[ijk}^{\pm a} \delta_l^b] - 2C_{[ijk}^{\pm b} \delta_l^a] ,\end{aligned}$$

where we have used eq. (A.3), in particular. Taking now  $a = m$  and  $b = n$  yields

$$\gamma_{mc}^{[ij} \gamma_{cn}^{kl]} = \delta_{mn} C^{ijkl} + 4C_{m[ijk} \delta_{l]n} + 4C_{n[ijk} \delta_{l]m} ,$$

For  $a = b = 8$  one finds

$$\gamma_{8c}^{[ij} \gamma_{c8}^{kl]} = -C_p^{[ij} C_p^{kl]} = C^{ijkl} ,$$

and taking  $a = m$ ,  $b = 8$  yields

$$\gamma_{mc}^{[ij} \gamma_{c8}^{kl]} = -C_{mp}^{[ij} C_p^{kl]} - 2C^{[ijk} \delta_m^{l]} = -4C^{[ijk} \delta_m^{l]} .$$

Thus we find

$$\gamma_{ab}^{ijkl} = \delta_{ab} C^{ijkl} + 4C_{a[ijk}^{\pm} \delta_{l]b} + 4C_{b[ijk}^{\pm} \delta_{l]a} , \quad (2.15)$$

where we have used eq. (A.2). In fact, we just proved the identity

$$-C_{ac}^{\pm [ij} C_{bc}^{\pm kl]} - 2C_{a[ijk}^{\pm} \delta_{l]b} - 2C_{b[ijk}^{\pm} \delta_{l]a} = \delta_{ab} C^{\pm ijkl} .$$

The explicit formulas for the gamma matrices will be used in the next section.

The matrices  $\gamma^{ij}$  represent the 21 generators  $J^{ij}$  of  $Spin(7)$  in its eight-dimensional spinor representation. One can extend the spinor representation of  $Spin(7)$  to the left handed or right handed spinor representation of  $SO(8)$  by adding the matrices  $\pm i\gamma^i$  [39, 41, 42]. By defining  $J^i = J^{i8}$ , the commutation relations of  $SO(8)$  can be written as

$$\begin{aligned}[J^i, J^j] &= 2J^{ij} , \\ [J^i, J^{mn}] &= 2\delta^{im} J^n - 2\delta^{in} J^m , \\ [J^{ij}, J^{kl}] &= 2\delta^{jk} J^{il} + 2\delta^{il} J^{jk} - 2\delta^{ik} J^{jl} - 2\delta^{jl} J^{ik} .\end{aligned} \quad (2.16)$$

The automorphism group  $G_2$  of octonions is a 14-dimensional subgroup of  $SO(7)$ . Under  $G_2$ , the adjoint representation of  $SO(7)$  decomposes as  $\underline{21} = \underline{14} + \underline{7}$ . We shall denote the generators of  $G_2$  as  $G^{ij}$ . One can choose a basis for  $G_2$  such that the generators  $G^{ij}$  can be expressed in terms of the generators  $J^{ij}$  of  $SO(7)$  in a simple form [41, 43]:

$$G^{ij} = \frac{1}{2}J^{ij} + \frac{1}{8}C^{ij}_{kl}J^{kl} . \quad (2.17)$$

Eq. (2.17) implies the linear relations

$$C_{ijk}G^{jk} = 0 , \quad (2.18)$$

and these are just the seven constraints that enforce the generators  $G^{ij}$  to span the 14-dimensional vector space [39]. Note also the related identities

$$C_{ijkl}G^{kl} = 2G_{ij} , \quad \text{and} \quad C_p^{[ij}G^{k]p} = 0 . \quad (2.19)$$

The remaining seven generators of  $SO(7)$  can be chosen as

$$A^i = \frac{1}{2}C^{ijk}J^{jk} , \quad (2.20)$$

They are associated with the seven-dimensional coset space  $SO(7)/G_2$ . Therefore, we arrive at the decomposition [43]

$$J^{ij} = \frac{4}{3}G^{ij} + \frac{1}{3}C^{ijk}A^k . \quad (2.21)$$

The  $G_2$  generators  $G^{ij}$  satisfy the commutation relations [43]:

$$[G^{ij}, G^{kl}] = 2\delta^{l[i}G^{j]k} - 2\delta^{k[i}G^{j]l} + \frac{1}{2}\left(C^{klm[i}G^{j]m} - C^{ijm[k}G^{l]m}\right) , \quad (2.22)$$

Furthermore, we have

$$[A^i, A^j] = -8G^{ij} + 2C^{ijk}A^k , \quad (2.23)$$

thus reflecting the fact that the coset space  $SO(7)/G_2$  is *not* a symmetric space. The symmetric space  $SO(8)/SO(7)$  can be identified with the round seven-sphere  $S^7$ . The space  $SO(7)/G_2$  can be considered as the seven-sphere with torsion, and we shall denote it in what follows as  $S_T^7$ .

The  $SO(8)$  generators can similarly be decomposed with respect to  $G_2$ ,

$$\underline{28} = \underline{14} + \underline{7} + \underline{7} , \quad (2.24)$$

with the generators  $J^i$  and  $A^i$  introduced above transforming in the seven-dimensional representation of  $G_2$ . In a  $G_2$  basis, the commutation relations of  $SO(8)$  take the form

$$\begin{aligned} [J^i, J^j] &= 2J^{ij} = \frac{8}{3}G^{ij} + \frac{2}{3}C^{ijk}A^k , \\ [A^i, J^j] &= -2C^{ijk}J^k , \\ [J^i, G^{kl}] &= \delta^{ik}J^l - \delta^{il}J^k + \frac{1}{2}C^{iklp}J^p , \\ [A^i, G^{kl}] &= \delta^{ik}A^l - \delta^{il}A^k + \frac{1}{2}C^{iklp}A^p , \end{aligned} \quad (2.25)$$

in addition to the commutation relations (2.16), (2.22) and (2.23).

The three eight-dimensional representations of  $SO(8)$  are in triality and the subgroup of  $SO(8)$  invariant under the triality mapping is  $G_2$  [39]. This is evident from the commutation relations of  $SO(8)$  in the  $G_2$  basis above. Note also the following additional identities:

$$\begin{aligned} C^{ij}_{kl}J^{kl} &= \frac{8}{3}G^{ij} - \frac{4}{3}C^{ijp}A^p , \\ J_{ij}J^{ij} &= \frac{16}{9}G_{ij}G^{ij} + \frac{2}{3}A_iA^i , \\ C_{ijkl}J^{ij}J^{kl} &= \frac{32}{9}G_{ij}G^{ij} - \frac{8}{3}A_iA^i = 2J_{ij}J^{ij} - 4A_iA^i . \end{aligned} \quad (2.26)$$

Another embedding of  $G_2$  into  $SO(7)$  that will be also usefull in the next sections was given in ref. [39] and later used in ref. [20]. In this embedding the fourteen generators

$M^A$  of  $G_2$  ( $A = 1, 2, \dots, 14$ ) are given as follows:

$$\begin{aligned}
M^1 &= \frac{1}{\sqrt{2}} (T^{41} + T^{36}) , & M^2 &= \frac{1}{\sqrt{6}} (T^{41} - T^{36} + 2T^{25}) , \\
M^3 &= \frac{1}{\sqrt{2}} (T^{31} - T^{46}) , & M^4 &= \frac{1}{\sqrt{6}} (T^{31} + T^{46} - 2T^{57}) , \\
M^5 &= \frac{1}{\sqrt{2}} (T^{21} - T^{76}) , & M^6 &= \frac{1}{\sqrt{6}} (T^{21} + T^{56} - 2T^{45}) , \\
M^7 &= \frac{1}{\sqrt{2}} (T^{71} + T^{26}) , & M^8 &= \frac{1}{\sqrt{6}} (T^{71} - T^{26} - 2T^{35}) , \\
M^9 &= \frac{1}{\sqrt{2}} (T^{24} - T^{73}) , & M^{10} &= \frac{1}{\sqrt{6}} (T^{24} + T^{73} + 2T^{15}) , \\
M^{11} &= \frac{1}{\sqrt{2}} (T^{74} + T^{23}) , & M^{12} &= \frac{1}{\sqrt{6}} (T^{74} - T^{23} - 2T^{65}) , \\
M^{13} &= \frac{1}{\sqrt{2}} (T^{43} - T^{16}) , & M^{14} &= \frac{1}{\sqrt{6}} (T^{43} + T^{16} + 2T^{27}) ,
\end{aligned} \tag{2.27}$$

where  $T^{ij}$  are  $SO(7)$  generators. For writing down the non-linear  $N = 7$  superconformal algebra (sect. 3), it is convenient to take the  $SO(7)$  generators here in the vector representation [20],

$$(T^{ij})_{kl} = -i (\delta_k^i \delta_l^j - \delta_l^i \delta_k^j) . \tag{2.28}$$

Finally, we give a few branching rules for the  $Spin(7)$  tensor products, namely

$$\underline{7} \times \underline{7} = \underline{1}_s + \underline{21}_a + \underline{27}_s , \quad \underline{8} \times \underline{7} = \underline{8} + \underline{48} , \quad \underline{8} \times \underline{8} = \underline{1}_s + \underline{7}_a + \underline{21}_a + \underline{35}_s , \tag{2.29}$$

where the  $\underline{8}$  stands for the 8-dimensional spinor representation. As far as  $G_2$  is concerned, the only decomposition to be relevant for us is given by

$$\underline{7} \times \underline{7} = \underline{1} + \underline{7} + \underline{14} + \underline{27} . \tag{2.30}$$

In Table I, we list some basic facts about  $G_2$ ,  $SO(7)$  and  $SO(8)$ .

Table I. The groups  $G_2$ ,  $SO(7)$  and  $SO(8)$

$G$	dimension	rank	dual Coxeter number
$G_2$	14	2	4
$SO(7)$	21	3	5
$SO(8)$	28	4	6

### 3 Exceptional non-linear superconformal algebras

In this section, we present the defining OPEs for the  $N = 8$  and  $N = 7$  non-linear superconformal algebras following ref. [20]. These algebras can be obtained either via a Drinfeld-Sokolov-type reduction from affine versions of the exceptional Lie superalgebras  $F(4)$  and  $G(3)$ , respectively [21], or by purely algebraic methods [19, 20]. Both algebras have generators of conformal dimension 2,  $3/2$  and 1 only. The  $N = 8$  algebra contains eight supercurrents  $S^M$  of conformal dimension  $3/2$ , and 21 symmetry currents of  $SO(7)$  under which the supercurrents transform in the *spinor* representation. The  $N = 7$  algebra has 7 supercurrents, and 14 symmetry currents of  $G_2$ . Both algebras contain a single generator of conformal dimension 2, and they are *completely* fixed by their field content and associativity (the Jacobi ‘identities’). Because of their non-linearity, the ‘vacuum-preserving’ algebra, generated by the modes  $L_0$ ,  $L_{\pm 1}$ ,  $S_{\pm 1/2}^M$  and  $T_0^A$ , is *not* finite. The OPEs to be given below are equivalent to the (anti)commutation relations of ref. [20].

### 3.1 Exceptional $N = 8$ superconformal algebra

The bosonic part of the  $N = 8$  algebra is a semi-direct product of the affine algebra  $\widehat{so(7)}_k$  of level  $k$  and the Virasoro algebra. The corresponding OPEs are given by

$$\begin{aligned} T(z) T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \\ T(z) T^{mn}(w) &\sim \frac{T^{mn}(w)}{(z-w)^2} + \frac{\partial T^{mn}(w)}{z-w}, \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} T^{mn}(z) T^{pq}(w) &\sim \frac{-i}{z-w} \{ \delta^{np} T^{mq}(w) + \delta^{mq} T^{np}(w) - \delta^{mp} T^{nq}(w) - \delta^{nq} T^{mp}(w) \} \\ &\quad + \frac{k}{(z-w)^2} \{ \delta^{mp} \delta^{nq} - \delta^{mq} \delta^{np} \}, \end{aligned} \quad (3.2)$$

where the adjoint of  $SO(7)$  is labeled by a pair of antisymmetric indices,  $m, n, \dots = 1, 2, \dots, 7$ . Compared to the previous section, we have normalised the affine spin-1 currents differently,  $J^{mn} = 2iT_0^{mn}$ , where  $T_0^{mn}$  is the zero-mode of  $T^{mn}(z)$ .<sup>5</sup>

Since the  $N = 8$  supercurrents  $S^M(z)$  transform in the spinor representation of  $SO(7)$  and have spin  $3/2$ , we have

$$\begin{aligned} T(z) S^M(w) &\sim \frac{\frac{3}{2}S^M(w)}{(z-w)^2} + \frac{\partial S^M(w)}{z-w}, \\ T^{mn}(z) S^M(w) &\sim \frac{-i}{2} \frac{\gamma_{MN}^{mn}}{(z-w)} S^N(w), \end{aligned} \quad (3.3)$$

The only non-trivial OPE's are the ones corresponding to the products of  $N = 8$  supersymmetry generators which read as follows:

$$\begin{aligned} S^M(z) S^N(w) &\sim \frac{8k(k+2)}{3(k+4)} \frac{\delta^{MN}}{(z-w)^3} + \frac{2T(w)}{z-w} \delta^{MN} - \frac{\delta^{MN}}{3(k+4)} \frac{:T^{mn}T^{mn}: (w)}{z-w} \\ &\quad + \frac{2i(k+2)}{3(k+4)} \gamma_{mn}^{MN} \left[ \frac{T^{mn}(w)}{(z-w)^2} + \frac{\partial T^{mn}(w)}{2(z-w)} \right] - \frac{1}{12(k+4)} \gamma_{mnpq}^{MN} \frac{:T^{mn}T^{pq}: (w)}{z-w}, \end{aligned} \quad (3.4)$$

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<sup>5</sup>The same normalisation convention was adopted in ref. [20].

where  $M, N = 1, \dots, 8$ , and  $i, j, \dots = 1, \dots, 7$ , as in the previous section. The above eq. (3.4) becomes more transparent and suitable for calculations after substituting the gamma matrices in terms of the octonionic structure constants (see sect. 2 and Appendix A). The relevant formulas are collected in Appendix B.

We have verified that all the Jacobi ‘identities’ are satisfied provided that the central charge  $c$  of the  $N = 8$  algebra is determined by the level  $k$  as follows:

$$c = c_8 \equiv 4k + \frac{6k}{k+4} \equiv \frac{2k(2k+11)}{k+4} , \quad (3.5)$$

in agreement with refs. [19, 20]. The identities (2.8) and (2.9) were crucial in checking the Jacobi ‘identities’ for the  $N=8$  algebra. Compared to Bowcock [20], our supersymmetry generator  $S^M$  above differs from his,  $S_B^M$ , by an overall scale factor, namely,  $S^M = \sqrt{\frac{2k+4}{2k+11}} S_B^M$ . It should be stressed that the very existence of such non-linear  $N = 8$  superconformal algebra is highly non-trivial, because several consistency requirements still have to be satisfied in the process of checking the Jacobi ‘identities’ when all free parameters are already fixed [20].

### 3.2 Exceptional $N = 7$ superconformal algebra

The exceptional  $N = 7$  non-linear superconformal algebra is similar to the  $N = 8$  algebra, with the gauge group  $G_2$  instead of  $SO(7)$ , and seven supercurrents. We shall denote the generators of  $G_2$  as  $G^A$  and not as  $M^A$  as we did in the previous section. The symbol  $M^A$  will be used for the seven-dimensional representation matrices of  $G_2$ ,  $A = 1, 2, \dots, 14$ , as given in eq. (2.27), providing the explicit embedding of  $G_2$  into  $SO(7)$  [39]. The matrices  $M^A$  satisfy the properties [20]

$$\begin{aligned} \text{tr} \left( M^A M^B \right) &= 2\delta^{AB} , \\ M_{ij}^A M_{kl}^A &= \frac{2}{3} (\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) - \frac{1}{3} C_{ijkl} . \end{aligned} \quad (3.6)$$



The bosonic OPEs of the  $N=7$  algebra are

$$\begin{aligned} T(z) T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \\ T(z) G^{ij}(w) &\sim \frac{G^{ij}(w)}{(z-w)^2} + \frac{\partial G^{ij}(w)}{z-w}. \end{aligned} \quad (3.7)$$

The  $G_2$  currents satisfy [43]

$$\begin{aligned} G^{ij}(z) G^{kl}(w) &\sim \frac{1}{z-w} \left\{ 2\delta^{l[i} G^{j]k}(w) - 2\delta^{k[i} G^{j]l}(w) \right. \\ &\quad \left. + \frac{1}{2} C^{klm[i} G^{j]m}(w) - \frac{1}{2} C^{ijm[k} G^{l]m}(w) \right\} \\ &\quad - \frac{k}{(z-w)^2} \left\{ \frac{3}{2} (\delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk}) + \frac{3}{4} C^{ijkl} \right\}. \end{aligned} \quad (3.8)$$

The generators  $G^{ij}$  of  $G_2$  are related to the generators  $G^A$  given in the previous section as

$$G_{ij} = \frac{3}{4i} M_{ij}^A G^A. \quad (3.9)$$

The seven supercurrents  $S^i(z)$  transform in  $\underline{7}$  of  $G_2$ , and satisfy the OPEs

$$\begin{aligned} T(z) S^i(w) &\sim \frac{\frac{3}{2} S^i(w)}{(z-w)^2} + \frac{\partial S^i(w)}{z-w}, \\ G^A(z) S^i(w) &\sim \frac{1}{z-w} M_{ij}^A S^j(w). \end{aligned} \quad (3.10)$$

The most important OPE's are again the ones defining the  $N = 7$  supersymmetry algebra, and they read as

$$\begin{aligned} S^i(z) S^j(w) &\sim \frac{k(3k+5)}{k+3} \frac{\delta^{ij}}{(z-w)^3} + \frac{3k+5}{k+3} M_{ij}^A \left[ \frac{G^A(w)}{(z-w)^2} + \frac{1}{2} \frac{\partial G^A(w)}{z-w} \right] \\ &\quad + \frac{\delta^{ij}}{z-w} \left[ 2T(w) - \frac{1}{k+3} : G^A G^A : (w) \right] \\ &\quad + \frac{3}{4(k+3)} \left[ M^A M^B + M^B M^A \right]^{ij} \frac{G^A G^B : (w)}{z-w}. \end{aligned} \quad (3.11)$$

We have verified that all the Jacobi ‘identities’ are indeed satisfied provided that the central charge is given by [19, 20]

$$c = c_7 \equiv \frac{9}{2}k + \frac{2k}{k+3} \equiv \frac{k(9k+31)}{2(k+3)}. \quad (3.12)$$

Again, it is fully consistent, in particular, with the results of Bowcock [20], after taking into account the rescaling  $S^i = \sqrt{\frac{9k+15}{9k+31}} S_B^i$ .

There is a confusion in the literature concerning the relationship between the two exceptional superconformal algebras. The  $N = 7$  non-linear algebra is *not* a subalgebra of the  $N=8$  non-linear algebra. This can be most easily seen in the limit  $c \rightarrow \infty$ , where both algebras linearise. The  $N = 8$  algebra contains the finite Lie superalgebra  $F(4)$  in that limit, which is its vacuum-preserving subalgebra. On the other hand, the corresponding subalgebra of the  $N = 7$  algebra is  $G(3)$ . If the  $N = 7$  algebra were a subalgebra of the  $N = 8$  one, then  $G(3)$  would have to be a subalgebra of  $F(4)$ . However, it is known that this is not the case. It follows from the fact that the smallest non-trivial representations of both Lie superalgebras are their adjoint representations. If  $G(3)$  were a subalgebra of  $F(4)$ , then this would imply that there be an 9-dimensional ( $\dim \text{ad } F(4) - \dim \text{ad } G(3) = 9$ ) non-trivial representation of  $G(3)$ , which does not exist [22].

## 4 Exceptional coset constructions

In this section, we shall investigate the possibility of realizing the exceptional superconformal algebras over certain special coset spaces  $G/H$ . We adopt here the following conventions: <sup>6</sup> we use the early Latin capital letters for  $G$  indices, and the early lower-case Latin letters for  $G/H$  indices,  $A, B, \dots = 1, \dots, \dim G$ , and  $a, b, \dots = \dim H + 1, \dots, \dim G$ .

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<sup>6</sup>The notation adopted here, in this section, for a *general* group  $G$  and its subgroup  $H$  slightly overlaps with our conventions in the previous sections and in what follows for the particular cosets. This should not lead to a confusion since we discuss the general and particular cosets separately in our paper.

Let  $\tilde{k}$  be (integer) level of affine algebra  $\widehat{G}$  realised in terms of (bosonic) currents  $\hat{J}^A(z)$ . The latter can be thought of as originating from the bosonic WZNW model on the group  $G$  [44, 2, 3], and they satisfy the OPE

$$\hat{J}^A(z)\hat{J}^B(w) = \frac{\tilde{k}/2}{(z-w)^2}\delta^{AB} + \frac{if^{ABD}\hat{J}^D(w)}{z-w} + : \hat{J}^A\hat{J}^B : (w) + \dots , \quad (4.1)$$

where  $if^{ABD}$  are the structure constants of  $G$ . Let's also introduce free fermions in the adjoint of  $G$ , with the OPE

$$\psi^A(z)\psi^B(w) = \frac{1/2}{z-w}\delta^{AB} + : \psi^A\psi^B : (w) + (z-w) : \partial\psi^A\psi^B : (w) + \dots . \quad (4.2)$$

These free fermions can be thought of as coming from the (1,0) *supersymmetric* (heterotic) WZNW model on the group  $G$  [45, 3].

The basic idea of coset construction is to construct generators of a given superconformal algebra in terms of the basic fields  $\hat{J}^a(z)$  and  $\psi^a(z)$  associated with a coset  $G/H$  [33, 2, 3]. Eqs. (4.1) and (4.2) obviously imply

$$\hat{J}^a(z)\hat{J}^b(w) = \frac{\tilde{k}/2}{(z-w)^2}\delta^{ab} + \frac{if^{abd}\hat{J}^d(w)}{z-w} + : \hat{J}^a\hat{J}^b : (w) + \dots , \quad (4.3)$$

and

$$\psi^a(z)\psi^b(w) = \frac{1/2}{z-w}\delta^{ab} + : \psi^a\psi^b : (w) + (z-w) : \partial\psi^a\psi^b : (w) + \dots . \quad (4.4)$$

The most general Ansatz for the supercurrents of any extended superconformal algebra over an arbitrary coset is given by

$$S^M(z) = 2\alpha(\tilde{k}) \left\{ h_{ab}^M \psi^a(z) \hat{J}^b(z) + \gamma(\tilde{k}) \xi_a^M \partial\psi^a(z) - \frac{2i}{3} \beta(\tilde{k}) \Gamma_{abc}^M : \psi^a\psi^b\psi^c : (z) \right\} , \quad (4.5)$$

where  $\alpha(\tilde{k})$ ,  $\gamma(\tilde{k})$  and  $\beta(\tilde{k})$  are some functions of the level  $\tilde{k}$ , while  $h_{ab}^M$ ,  $\xi_a^M$  and  $\Gamma_{abc}^M$  are some tensors, the latter being totally antisymmetric with respect to its subscripts.<sup>7</sup>

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<sup>7</sup>No symmetry properties are à priori assumed for  $h_{ab}^M$  and  $\xi_a^M$ .

The Ansatz (4.5) is dictated by dimensional reasons. The tensors  $h_{ab}^M$ ,  $\xi_a^M$  and  $\Gamma_{abc}^M$  have to be consistent with the transformation properties of the conformal fields in the Ansatz (4.5). For example, the ‘background charge’ terms proportional to  $\partial\psi^a$  can only contribute when there exists a mixed tensor  $\xi_a^M$  which is invariant under the  $SO(7)$  transformations in the N=8 case or under the  $G_2$  transformations in the N=7 case. That is only possible if this tensor is proportional to the delta-symbol, which implies, in particular when  $\gamma \neq 0$ , that some free fermions  $\psi^a$  should transform in the *same*  $SO(7)$  or  $G_2$  representation as the N=8 or N=7 supercurrents, respectively.

The tensors  $h_{ab}^M$  can be geometrically interpreted as the generalised complex structures on the coset in question, whereas  $\Gamma_{abc}^M$  as the generalized torsion coefficients. The tensor  $\xi_a^M$  represents the background charges (see subsect. 4.2 for a non-trivial example).

The Ansatz (4.5) is supposed to be completely fixed by the superconformal algebra. This is known to be the case for the superconformal algebras with  $N \leq 4$  [33, 34, 35, 36, 37, 38], and it is expected to be the case in general. In fact, the resulting constraints usually lead to an overdetermined system of equations, so that it is highly non-trivial whether the equations are really consistent and lead to a solution when the number of supersymmetries is larger than four. As we shall show below, there exist very few consistent solutions to these constraints in the case of the exceptional  $N = 7$  and  $N = 8$  extended superconformal algebras.

For the unitary representations to be constructed via the coset space method, the coefficients on the r.h.s. of eq. (4.5) for the supercurrents have to be real, if the fermions are normalised as above and the currents  $\hat{J}^a$  are hermitian.

It is straightforward to calculate the OPE that the supercurrents (4.5) satisfy. We

find

$$\begin{aligned}
(4\alpha^2)^{-1} S^M(z) S^N(w) \sim & \frac{1}{(z-w)^3} \left[ \frac{\tilde{k}}{4} h_{ab}^M h_{ab}^N + \frac{1}{3} \beta^2 \Gamma_{abc}^M \Gamma_{abc}^N - \gamma^2 \xi_a^M \xi_a^N \right] \\
& + \frac{1}{(z-w)^2} \left[ \frac{i}{2} h_{ab}^M h_{ad}^N f^{bdD} \hat{J}^D + \frac{\tilde{k}}{2} h_{ab}^M h_{cb}^N : \psi^a \psi^c : \right. \\
& + 2\beta^2 \Gamma_{abc}^M \Gamma_{abd}^N : \psi^c \psi^d : + \gamma \frac{1}{2} (h_{cb}^M \xi_c^N - h_{cb}^N \xi_c^M) \hat{J}^b \\
& - 2i\beta\gamma (\Gamma_{abc}^M \xi_a^N - \Gamma_{abc}^N \xi_a^M) : \psi^b \psi^c : \left. \right] (w) \\
& + \frac{1}{z-w} \left[ + \frac{1}{2} h_{ab}^M h_{ad}^N (: \hat{J}^b \hat{J}^d : + \frac{1}{2} i f^{bdD} \partial \hat{J}^D) + \frac{1}{2} \gamma h_{ab}^M \xi_a^N \partial \hat{J}^b \right. \\
& + 2\beta^2 \Gamma_{abf}^M \Gamma_{abg}^N : \partial \psi^f \psi^g : - i\beta (\Gamma_{abc}^M h_{cf}^N + \Gamma_{abc}^N h_{cf}^M) \hat{J}^f : \psi^a \psi^b : \\
& - 4i\beta\gamma \Gamma_{abc}^M \xi_a^N : \psi^b \partial \psi^c : + i h_{ab}^M h_{cd}^N f^{bdD} \hat{J}^D : \psi^a \psi^c : \\
& \left. + \frac{\tilde{k}}{2} h_{ab}^M h_{cb}^N : \partial \psi^a \psi^c : - 2\beta^2 \Gamma_{abc}^M \Gamma_{afg}^N : \psi^b \psi^c \psi^f \psi^g : \right] (w) .
\end{aligned} \tag{4.6}$$

As far as the  $N = 8$  algebra is concerned, comparing the residues at the  $(z-w)^{-3}$ ,  $(z-w)^{-2}$  and  $(z-w)^{-1}$  poles in eqs. (3.4) and (4.6), respectively, yields

$$\frac{8k(k+2)}{3(k+4)} \delta^{MN} = \alpha^2 \left[ \tilde{k} h_{ab}^M h_{ab}^N + \frac{4}{3} \beta^2 \Gamma_{abc}^M \Gamma_{abc}^N - 4\gamma^2 \xi_a^M \xi_a^N \right] , \tag{4.7a}$$

$$\begin{aligned}
\frac{2i(k+2)}{3(k+4)} \gamma_{mn}^{MN} T^{mn}(z) = & \alpha^2 \left[ 2i h_{ab}^M h_{ad}^N f^{bdD} \hat{J}^D + 4\gamma h_{ab}^{[M} \xi_a^{N]} \hat{J}^b - 16i\beta\gamma \Gamma_{abc}^{[M} \xi_a^{N]} : \psi^b \psi^c : \right. \\
& \left. + 2\tilde{k} h_{ab}^M h_{cb}^N : \psi^a \psi^c : + 8\beta^2 \Gamma_{abc}^M \Gamma_{abd}^N : \psi^c \psi^d : \right] (z) ,
\end{aligned} \tag{4.7b}$$

and

$$\begin{aligned}
& \left[ 2T(z) - \frac{1}{3(k+4)} : T^{mn} T^{mn} : (z) \right] \delta^{MN} + \frac{i(k+2)}{3(k+4)} \gamma_{mn}^{MN} \partial T^{mn}(z) \\
& - \frac{1}{12(k+4)} \gamma_{mnpq}^{MN} : T^{mn} T^{pq} : (z) = \alpha^2 \left[ 2h_{ab}^M h_{ad}^N (: \hat{J}^b \hat{J}^d : + \frac{1}{2} i f^{bdD} \partial \hat{J}^D) (z) \right. \\
& + 4i h_{ab}^M h_{cd}^N f^{bdD} \hat{J}^D : \psi^a \psi^c : (z) - 4i\beta (\Gamma_{abc}^M h_{cf}^N + \Gamma_{abc}^N h_{cf}^M) \hat{J}^f : \psi^a \psi^b : (z) \\
& \left. + 2\tilde{k} h_{ab}^M h_{cb}^N : \partial \psi^a \psi^c : (z) + 8\beta^2 \Gamma_{abf}^M \Gamma_{abg}^N : \partial \psi^f \psi^g : (z) + 2\gamma h_{ab}^M \xi_a^N \partial \hat{J}^b (z) \right]
\end{aligned}$$

$$-16i\beta\gamma\Gamma_{abc}^M\xi_a^N : \psi^b\partial\psi^c : (z) - 8\beta^2\Gamma_{abc}^M\Gamma_{afg}^N : \psi^b\psi^c\psi^f\psi^g : (z) \Big] . \quad (4.7c)$$

where  $M, N = 1, 2, \dots, 8$ .

Quite similar equations appear in the case of the  $N=7$  algebra. We find

$$\frac{k(3k+5)}{k+3}\delta^{mn} = \alpha^2 \left[ \tilde{k}h_{ab}^mh_{ab}^n + \frac{4}{3}\beta^2\Gamma_{abc}^m\Gamma_{abc}^n - 4\gamma^2\xi_a^m\xi_a^n \right] , \quad (4.8a)$$

$$\begin{aligned} \frac{3k+5}{k+3} (M^A)^{mn} G^A(z) = & \alpha^2 \left[ 2ih_{ab}^mh_{ad}^nf^{bdD}\hat{J}^D(z) + 4\gamma h_{ab}^{[m}\xi_a^{n]} \hat{J}^b - 16i\beta\gamma\Gamma_{abc}^{[m}\xi_a^{n]} : \psi^b\psi^c : \right. \\ & \left. + 2\tilde{k}h_{ab}^mh_{cb}^n : \psi^a\psi^c : (z) + 8\beta^2\Gamma_{abc}^m\Gamma_{abd}^n : \psi^c\psi^d : (z) \right] , \end{aligned} \quad (4.8b)$$

and

$$\begin{aligned} & \left[ 2T(z) - \frac{1}{k+3} : G^AG^A : (z) \right] \delta^{mn} + \frac{3k+5}{2(k+3)} (M^A)^{mn} \partial G^A(z) \\ & + \frac{1}{4(k+3)} (M^AM^B + M^BM^A)^{mn} : G^AG^B : (z) = \alpha^2 \left[ 2h_{ab}^mh_{ad}^n \left( : \hat{J}^b\hat{J}^d : + \frac{1}{2}if^{bdD}\partial\hat{J}^D \right) (z) \right. \\ & + 2\tilde{k}h_{ab}^mh_{cb}^n : \partial\psi^a\psi^c : (z) + 4ih_{ab}^mh_{cd}^nf^{bdD}\hat{J}^D : \psi^a\psi^c : (z) + 2\gamma h_{ab}^m\xi_a^n \partial\hat{J}^b(z) \\ & - 4i\beta \left( \Gamma_{abc}^mh_{cf}^n + \Gamma_{abc}^n h_{cf}^m \right) \hat{J}^f : \psi^a\psi^b : (z) + 8\beta^2\Gamma_{abf}^m\Gamma_{abg}^n : \partial\psi^f\psi^g : (z) \\ & \left. - 16i\beta\gamma\Gamma_{abc}^M\xi_a^N : \psi^b\partial\psi^c : (z) - 8\beta^2\Gamma_{abc}^M\Gamma_{afg}^N : \psi^b\psi^c\psi^f\psi^g : (z) \right] , \end{aligned} \quad (4.8c)$$

where  $m, n = 1, 2, \dots, 7$ , and  $A = 1, 2, \dots, 14$ .

For both  $N = 7$  and  $N = 8$  algebras, eq. (a) determines, in particular, the level  $k$  of the algebra, eq. (b) determines the affine currents of the algebra, while eq. (c) determines the stress tensor. Furthermore, for each equation, there are the complicated non-linear consistency conditions on the unknown constant tensors  $h$ ,  $\xi$  and  $\Gamma$ , and the unknown coefficients  $\alpha$ ,  $\beta$  and  $\gamma$ . For example, as far as the  $(M, N)$ -*symmetric* simple-pole contributions on the r.h.s. of eq. (4.7c) for the  $N = 8$  algebra are concerned, the coset current  $(\hat{J}^a)$ -dependent terms among them are given by

$$h_{ab}^{(M)}h_{ad}^{(N)} : \hat{J}^b\hat{J}^d : - 4i\beta\Gamma_{abc}^{(M)}h_{cf}^{(N)}\hat{J}^f : \psi^a\psi^b : + \gamma h_{ab}^{(M)}\xi_a^{(N)}\partial\hat{J}^b . \quad (4.9)$$

They can only contribute to the trace ( $\delta^{MN}$ -dependent) terms, according to eq. (4.7c). Moreover, the term quadratic in the coset currents has to be diagonal (i.e. of Sugawara form), since it is going to contribute to the stress tensor  $T$ . Therefore, we conclude that

$$h_{ab}^{(M} h_{ad}^{N)} \sim \delta^{MN} \delta_{bd} , \quad \Gamma_{abc}^{(M} h_{cf}^{N)} \sim \delta^{MN} , \quad \text{and} \quad h_{ab}^M \xi_a^N \sim \delta^{MN} , \quad (4.10)$$

where, in the last equation, we have also taken into account the restrictions coming from the antisymmetric terms in eq. (4.7c) too. The conditions (4.10) are highly restrictive since, in addition, all the tensors  $h$ ,  $\Gamma$  and  $\xi$  are to be consistent with the transformation properties of the both sides of eq. (4.5).

The first equation (4.10) implies that  $h_{ab}^m$  must be the seven-dimensional  $8 \times 8$  gamma matrices and  $h^8 \sim \mathbf{1}_8$  if we assume that the indices  $a, b, \dots$  take values in an irreducible representation of  $SO(7)$  (by Schur's lemma).<sup>8</sup> The second equation (4.10) then becomes equivalent to the relation

$$\Gamma_{abc}^m = i h_{cd}^m \Gamma_{abd}^8 . \quad (4.11)$$

It is not difficult to verify that eq. (4.11) does not have a non-trivial solution for  $\Gamma_{abc}^M$  which would be totally antisymmetric with respect to its subscript indices, as required. Hence, we have to conclude that either all  $\hat{J}^m$  or all  $\Gamma^M$  have to vanish. Similar conclusions follow for the  $N=7$  case too (see subsect. 4.2).

Thus, the coset we are looking for, if any, should be  $(7+1)$ -dimensional, the seven-dimensional space being represented by a seven-sphere (the only parallelizable coset space in seven dimensions). Indeed, the very existence of the exceptional  $N=8$  and  $N=7$  algebras crucially depends upon unique properties of gamma matrices in seven dimensions, which are related to octonions. One should therefore have expected that

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<sup>8</sup>The naive solution – the eight-dimensional  $16 \times 16$  gamma matrices — is ruled out because it leads to  $SO(8)$  gauge invariance instead of  $SO(7)$  required.

the coset spaces in question are to be the ones given by various symmetry groups of octonions. The naive candidates for such cosets would be  $SO(8)/SO(7)$  for the  $N = 8$  algebra, which corresponds to the round seven-sphere  $S^7$ , and  $SO(7)/G_2$  for the  $N = 7$  algebra, which corresponds to the  $S^7_T$  with torsion. However, it is not difficult to convince oneself that these naive coset spaces can *not* be the right ones. For the  $N = 8$  algebra, we need supercurrents that transform in the spinor representation of  $SO(7)$ , which can not be obtained from the currents and fermions on  $S^7$  that transform in the vector representation of  $SO(7)$ , according to eq. (2.29). For the  $N = 7$  algebra, the naive guess fails due to the fact that  $SO(7)/G_2$  is not a symmetric space, which leads to some unwanted terms in the OPEs. Remarkably enough, a simple extension of the naive coset spaces by a  $U(1)$  factor, i.e. adding a circle or ‘1-sphere’  $S^1$ , leads to the consistent solutions for the above constraints (4.7) and (4.8), as we are going to demonstrate in the rest of the paper. Adding the  $U(1)$  current  $J(z)$  is equivalent to introducing a scalar field  $\phi(z)$  since  $J \sim \partial\phi$  up to a normalisation constant (*cf.* ref. [46] where the free-field representations for the orthogonal series of the Bershadsky-Knizhnik non-linear  $N$ -extended superconformal algebras were constructed).

#### 4.1 A construction of the exceptional $N = 8$ superconformal algebra over the coset space $SO(8) \times U(1)/SO(7)$

Our starting point is the affine algebra  $\widehat{so(8)}_{\hat{k}} \oplus \widehat{u(1)}$ , defined by the OPEs

$$\begin{aligned} \hat{J}^{ab}(z)\hat{J}^{cd}(w) \sim & \frac{2}{z-w} \left\{ \delta^{bc}\hat{J}^{ad}(w) + \delta^{ad}J^{bc}(w) - \delta^{ac}\hat{J}^{bd}(w) - \delta^{bd}\hat{J}^{ac}(w) \right\} \\ & - \frac{4\hat{k}}{(z-w)^2} \left\{ \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc} \right\} , \end{aligned} \quad (4.12)$$

and

$$\hat{J}^8(z)\hat{J}^8(w) \sim \frac{\hat{k}_1/2}{(z-w)^2} , \quad (4.13)$$



where  $a, b, \dots = 1, 2, \dots, 8$ , and  $\hat{k}_1$  is a normalisation parameter of the  $U(1)$  current  $\hat{J}^8$ . The latter can be represented in terms of a scalar field,  $\hat{J}^8 \equiv i\sqrt{\hat{k}_1}/2 \partial\phi$ .

Because of eq. (4.12), the currents  $\hat{J}^m = \pm i\hat{J}^{m8}$ ,  $m = 1, \dots, 7$ , satisfy the OPE

$$\hat{J}^m(z)\hat{J}^n(w) = \frac{4\hat{k}}{(z-w)^2}\delta^{mn} + \frac{2\hat{J}^{mn}(w)}{z-w} + : \hat{J}^m \hat{J}^n : (w) + \dots \quad (4.14)$$

Associated with the  $U(1)$  factor is an additional free fermionic field  $\psi^8(z)$ , with the OPE

$$\psi^8(z)\psi^8(w) = \frac{1/2}{z-w} + (z-w) : \partial\psi^8\psi^8 : (w) + \dots \quad (4.15)$$

This field  $\psi^8$  together with the fermions  $\psi^m$  form an 8-dimensional column  $\psi^a$ , with the OPE (4.4). The currents  $\hat{J}^m$  transform in  $\underline{7}$  of  $SO(7)$ , while  $\hat{J}^8$  is a singlet. The  $\psi^a$  will transform in the  $\underline{8}$  of a  $Spin(7)$  algebra to be constructed from fermion bilinears.

Being applied to the particular coset space  $SO(8) \times U(1)/SO(7)$ , our general Ansatz (4.5) for the  $N = 8$  supercurrents can be simplified to

$$\begin{aligned} S^m(z) &= 2\alpha \left\{ \gamma_{an}^m \psi^a(z) \hat{J}^n(z) + i\psi^m(z) \hat{J}^8(z) + \gamma \partial\psi^m(z) - \frac{2i}{3} \beta \Gamma_{abc}^m : \psi^a \psi^b \psi^c : (z) \right\} , \\ S^8(z) &= 2\alpha \left\{ -i\psi^n(z) \hat{J}^n(z) - i\psi^8(z) \hat{J}^8(z) + \gamma \partial\psi^8(z) - \frac{2i}{3} \beta \Gamma_{abc}^8 : \psi^a \psi^b \psi^c : (z) \right\} , \end{aligned} \quad (4.16)$$

where the parameter  $\gamma = \gamma(\hat{k})$  plays the role of a background charge. To simplify the structure of our Ansatz (4.16) even further, we represent the generalised complex structures in terms of the ‘extended’ gamma matrices to be defined as

$$h_{ab}^M = \gamma_{ab}^M , \quad M = 1, 2, \dots, 8 , \quad \text{with} \quad \gamma^8 \equiv -i\mathbf{1}_8 , \quad (4.17)$$

These matrices satisfy the identities

$$\gamma_{ab}^8 \gamma_{ab}^8 = -8 , \quad \gamma_{ab}^m \gamma_{ab}^8 = 0 , \quad \gamma_{ab}^M \gamma_{ab}^N = -8\delta^{MN} , \quad (4.18a)$$

and

$$\begin{aligned}\gamma_{na}^m \psi^a \hat{J}^m + \gamma_{na}^8 \psi^a \hat{J}^8 &= \gamma_{ab}^n \psi^a \hat{J}^b , \\ \gamma_{8a}^m \psi^a \hat{J}^m + \gamma_{8a}^8 \psi^a \hat{J}^8 &= \gamma_{ab}^8 \psi^a \hat{J}^b ,\end{aligned}\tag{4.18b}$$

and allow us to rewrite eq. (4.16) as follows:

$$S^M(z) = 2\alpha(\hat{k}) \left\{ \gamma_{ab}^M \psi^a(z) \hat{J}^b(z) + \gamma(\hat{k}) \partial \psi^M(z) - \frac{2i}{3} \beta(\hat{k}) \Gamma_{abc}^M : \psi^a \psi^b \psi^c : (z) \right\} . \tag{4.19}$$

The only choice for the generalised torsion coefficients  $\Gamma_{abc}^M$  on a seven-sphere is

$$\Gamma_{mnp}^8 = \bar{A} C_{mnp} , \quad \Gamma_{npq}^m = \bar{B} C_{npq}^m , \quad \Gamma_{np8}^m = \bar{C} C_{np}^m , \tag{4.20}$$

where the coefficients  $\bar{A}$ ,  $\bar{B}$  and  $\bar{C}$  are at our disposal.

We thus reduce the problem of a coset space realization for the  $N=8$  algebra to finding a solution for the coefficients  $(\alpha, \beta, \gamma, \hat{k}_1, \bar{A}, \bar{B}, \bar{C})$  from a consistency of the OPE (4.6) in terms of our Ansatz supercurrents (4.19) with the OPEs of the  $N = 8$  algebra. In particular, the r.h.s. of eq. (4.7a) must reproduce  $\delta^{MN}$  and determine the level  $k$ . Eq. (4.7b) can be used to determine the  $SO(7)$  affine currents  $T^{mn}(z)$  of the  $N = 8$  algebra: taking  $M = m$  and  $N = n$ , where  $m, n = 1, 2, \dots, 7$ , we obtain the expressions for  $T^{mn}(z)$  and, hence,  $C_{mn}^p T^{mn}(z)$ , after using eqs. (2.14), (4.18) and (4.20), and the identities of Appendix A. On the other hand, taking  $M = m$  and  $N = 8$  in the same eq. (4.7b), we can directly calculate  $C_{mn}^p T^{mn}(z)$ . Both results must agree, and this gives us a non-trivial consistency relation. The simple-pole contributions of eq. (4.7c) produce three equations: the trace part proportional to  $\delta^{MN}$  determines the stress tensor  $T$  of the  $N = 8$  algebra, whereas the antisymmetric part and the traceless symmetric part proportional to  $\gamma_{mn}^{MN}$  and  $\gamma_{mnpq}^{MN}$ , respectively, yield the consistency relations for the already determined operators  $\partial T^{mn}$  and  $: T^{mn} T^{pq} :$ .

The most severe restrictions come out of the symmetric traceless part of the simple-pole terms. First of all, the coset current ( $\hat{J}^M$ -dependent) contributions have to cancel,

since they are obviously not allowed to contribute to a bilinear in the  $SO(7)$  currents. There are two different types of such unwanted terms on the r.h.s. of eq. (4.7c). First, the  $\gamma$ -dependent contribution contains the term  $\gamma \frac{1}{2} \gamma_{Nb}^M \partial \hat{J}^b$  which has to be proportional to  $\delta_N^M$ . However, it is only possible if  $\gamma = 0$  unless  $\hat{J}^a \neq 0$ . Hence, we have

$$\gamma = 0 \ , \quad \text{or} \quad \hat{J}^m = \hat{J}^8 = 0 \ . \quad (4.21)$$

Second, there are different unwanted terms of the form

$$2\beta(C_{mpt}\hat{J}^p\psi^t + \hat{J}^m\psi^8)\psi^r + (m \leftrightarrow r) \ , \quad (4.22a)$$

in the OPE for  $S^m(z)S^r(w)$ , and that of the form

$$2\beta(C_{rnp}\psi^8\psi^p + \psi^n\psi^r)\hat{J}^n \ , \quad (4.22b)$$

in the OPE for  $S^8(z)S^r(w)$ . These unwanted terms vanish if and only if

$$\beta = 0 \ , \quad \text{or} \quad \hat{J}^m = 0 \ . \quad (4.23)$$

To this end, we examine in detail both non-trivial possibilities:

- (i)  $\hat{J}^m \neq 0$  , and  $\beta = \gamma = 0$ ,
- (ii)  $\hat{J}^m = \gamma \hat{J}^8 = 0$  , and  $\beta \neq 0$ .

(i). This case corresponds to having no trilinear fermions in the Ansatz ( $\beta = 0$ ), as well as no background charge ( $\gamma = 0$ ). From eq. (4.7b) we find

$$\begin{aligned} \frac{i(k+2)}{6(k+4)}T^{mn} &= \alpha^2 \left\{ -\hat{J}^{mn} - 4\hat{k}\psi^m\psi^n + 4\hat{k}C^{mn}{}_p\psi^p\psi^8 + \frac{1}{2}(\hat{k} - \frac{1}{8}\hat{k}_1)C^{mn}{}_{pq}\psi^p\psi^q \right\} \ , \\ \frac{i(k+2)}{6(k+4)}C^m{}_{pq}T^{pq} &= \alpha^2 \left\{ -C^m{}_{pq}\hat{J}^{pq} + 4\hat{k}C^m{}_{pq}\psi^p\psi^q + \left(4\hat{k} + \frac{1}{2}\hat{k}_1\right)\psi^m\psi^8 \right\} \ . \end{aligned} \quad (4.24)$$

Multiplying the first line of this equation by  $C^p{}_{mn}$  and comparing the result with the second line yields two equations for the coefficients at the fermionic terms. Fortunately,

these two equations turn out to be the same if we set:

$$\hat{k}_1 = 40\hat{k} . \quad (4.25)$$

The first line of eq. (4.24) now takes the form

$$T^{mn} = \frac{i6(k+4)}{k+2} \alpha^2 \left\{ \hat{J}^{mn} + 2\hat{k} \left( 2\psi^m \psi^n - 2C^{mn}{}_p \psi^p \psi^8 + C^{mn}{}_{pq} \psi^p \psi^q \right) \right\} , \quad (4.26)$$

where we can recognize the fermionic  $Spin(7)$  generators, because of the identity

$$2\psi^{[m} \psi^{n]} + C^{mn}{}_{pq} \psi^p \psi^q - 2C^{mn}{}_p \psi^p \psi^8 = \bar{\psi} \tilde{\gamma}^{mn} \psi , \quad (4.27)$$

in terms of the Majorana spinor  $\psi$ ,  $\bar{\psi} = \psi^\dagger = \psi^T$ , with the components  $\psi^a$ , if we take the lower sign in eq. (2.14). Note also the related identities

$$\begin{aligned} C^p{}_{mn} \bar{\psi} \tilde{\gamma}^{mn} \psi &= -2 \left( C^p{}_{mn} \psi^m \psi^n + 6\psi^p \psi^8 \right) , \\ C^{pq}{}_{mn} \bar{\psi} \tilde{\gamma}^{mn} \psi &= 8 \left( \psi^p \psi^q + C^{pq}{}_m \psi^m \psi^8 \right) . \end{aligned} \quad (4.28)$$

Eq. (4.7a) consistently produces  $\delta^{MN}$  on the r.h.s. (the identities (4.18) play an important role here!) and determines  $\alpha$ ,

$$\alpha^2 = -\frac{k(k+2)}{36\hat{k}(k+4)} , \quad (4.29)$$

while eq. (4.26) can now be rewritten in an explicitly  $SO(7)$ -covariant form,

$$T^{mn}(z) = -\frac{ik}{6\hat{k}} \left\{ \hat{J}^{mn}(z) + 2\hat{k} : \bar{\psi} \tilde{\gamma}^{mn} \psi : (z) \right\} . \quad (4.30)$$

Eq. (4.30) determines the level  $k$  of the  $N = 8$  algebra, by comparing the double-pole contributions on the both sides of the OPE defining the  $\widehat{SO(7)}$  algebra that these currents satisfy. A direct calculation gives

$$k = \frac{k^2}{9\hat{k}^2} \left( \hat{k} + 4\hat{k}^2 \cdot 1 \right) , \quad \text{or} \quad k = \frac{9\hat{k}}{1 + 4\hat{k}} . \quad (4.31)$$

As a result, we get the following simple expression for the  $Spin(7)$  current  $T^{mn}$ :

$$T^{mn}(z) = -\frac{3i}{2(1+4\hat{k})} \left\{ \hat{J}^{mn}(z) + 2\hat{k} : \bar{\psi} \tilde{\gamma}^{mn} \psi : (z) \right\} . \quad (4.32)$$

Taking the trace in eq. (4.7c) with respect to the indices  $M = N$ , and using  $\delta^{MM} = 8$  and the obvious properties of the gamma matrices:  $\text{tr}(\gamma_{mn}) = \text{tr}(\gamma_{mnpq}) = 0$ , we find

$$\begin{aligned} T - \frac{1}{6(k+4)} : T^{mn} T^{mn} : = & \alpha^2 \left[ - : \hat{J}^a \hat{J}^a : + \frac{1}{8} (7\tilde{k} + 40\hat{k}) : \psi^a \partial \psi^a : - \hat{J}^{mn} : \psi^m \psi^n : \right. \\ & \left. - \frac{1}{2} C_{mnpq} \hat{J}^{mn} : \psi^p \psi^q : + C_{mnp} \hat{J}^{mn} : \psi^p \psi^8 : \right] \end{aligned} \quad (4.33)$$

where eqs. (2.9), (2.11) and (4.17), the book-keeping definition  $\hat{J}^a \equiv (\hat{J}^m, \hat{J}^8)$ , as well as the identities of Appendix A, have been used.

Next, making use of the definitions

$$\begin{aligned} \gamma_{ab}^m &= i \left( C_{ab}^m + \delta_a^m \delta_b^8 - \delta_b^m \delta_a^8 \right) , & \gamma^{mn} &= \gamma^{[m} \gamma^{n]} , \\ \tilde{\gamma}_{ab}^m &= i \left( C_{ab}^m - \delta_a^m \delta_b^8 + \delta_b^m \delta_a^8 \right) , & \tilde{\gamma}^{mn} &= \tilde{\gamma}^{[m} \tilde{\gamma}^{n]} , \end{aligned} \quad (4.34)$$

and the related identities (4.27) together with the identity

$$C_{mnpq} : \psi^m \psi^n \psi^p \psi^q : + 4C_{mnp} : \psi^m \psi^n \psi^p \psi^8 : = \frac{1}{2} : (\bar{\psi} \tilde{\gamma}^{mn} \psi) (\bar{\psi} \tilde{\gamma}^{mn} \psi) : , \quad (4.35)$$

we can rewrite eq. (4.33) in a compact and elegant form,

$$T - \frac{1}{6(k+4)} : T^{mn} T^{mn} : = \alpha^2 \left[ - : \hat{J}^a \hat{J}^a : + \frac{1}{8} (7\tilde{k} + 40\hat{k}) : \bar{\psi} \partial \psi : - \frac{1}{2} \hat{J}^{mn} : (\bar{\psi} \tilde{\gamma}^{mn} \psi) : \right] . \quad (4.36)$$

Eq. (4.36) is also explicitly  $SO(7)$ -covariant, which is important for the consistency of our calculations. The coefficients in eq. (4.36) follow from eqs. (4.14), (4.29) and (4.31):

$$\tilde{k} = 8\hat{k} , \quad \alpha^2 = -\frac{2+17\hat{k}}{4(1+4\hat{k})(4+25\hat{k})} . \quad (4.37)$$

Hence, we get from eq. (4.36) that the  $N = 8$  stress tensor is given by

$$T = \frac{1}{8(1 + \hat{k})(4 + 25\hat{k})} \left\{ -3 : \hat{J}^{mn} \hat{J}^{mn} : + 2(2 + 17\hat{k}) : \hat{J}^a \hat{J}^a : - 24\hat{k}(2 + 17\hat{k}) : \bar{\psi} \partial \psi : \right. \\ \left. + (2 + 5\hat{k}) \hat{J}^{mn} : (\bar{\psi} \tilde{\gamma}^{mn} \psi) : - 12\hat{k}^2 : (\bar{\psi} \tilde{\gamma}^{mn} \psi) (\bar{\psi} \tilde{\gamma}^{mn} \psi) : \right\} . \quad (4.38)$$

where we have used eq. (4.33). It is straightforward to check the rest of the  $N = 8$  algebra OPEs. In particular, all the equations (4.7) now become identities.

Since the level of an affine Lie algebra based on a compact Lie group must be a positive integer for unitary highest-weight representations, eq. (4.31) implies that we must consider either non-highest-weight-type unitary representations or non-unitary representations of  $\widehat{SO}(8)$ , in order to have a positive integer  $k$ , in general. The only exception exists when  $\hat{k} = 2$ , which yields  $k = 2$  also. According to eq. (3.5), the corresponding central charge is given by

$$c_8 = 10 . \quad (4.39)$$

The full list of the  $N = 8$  algebra generators in the case of  $\hat{k} = k = 2$ ,  $c_8 = 10$ , reads:

$$T^{mn} = -\frac{i}{6} \left\{ \hat{J}^{mn} + 4 \bar{\psi} \tilde{\gamma}^{mn} \psi \right\} , \\ S^m = \frac{2i}{3\sqrt{6}} \gamma_{ab}^m \psi^a \hat{J}^b , \quad S^8 = \frac{2}{3\sqrt{6}} \psi^a \hat{J}^a , \\ T = \frac{1}{18} : \hat{J}^a \hat{J}^a : - \frac{1}{432} : \hat{J}^{mn} \hat{J}^{mn} : - \frac{4}{3} : \bar{\psi} \partial \psi : \\ + \frac{1}{108} \hat{J}^{mn} : (\bar{\psi} \tilde{\gamma}^{mn} \psi) : - \frac{1}{27} : (\bar{\psi} \tilde{\gamma}^{mn} \psi) (\bar{\psi} \tilde{\gamma}^{mn} \psi) : . \quad (4.40)$$

We should note however that the choice  $\hat{k} = 2$  is not consistent with the defining (anti)-commutation relations of the  $N = 8$  algebra since by repeated commutation of the current  $T^{mn} \sim (\hat{J}^{mn} + 4 \bar{\psi} \tilde{\gamma}^{mn} \psi)$  with itself one generates currents of the form  $(\hat{J}^{mn} + 4^l \bar{\psi} \tilde{\gamma}^{mn} \psi)$ , where  $l = 1, 2, 3, \dots$ . If we choose  $\hat{k} = 2$  we will have to extend the algebra to a larger one.

Since the affine current  $T^{mn}$  of the  $N = 8$  algebra in eq. (4.30) is a linear combination of the bosonic and fermionic contributions,  $-(i/2)\hat{J}^{mn}$  and  $-(i/2)\bar{\psi}\tilde{\gamma}^{mn}\psi$ , all having the same (classical) normalisation,  $T^{mn}$  would be *precisely* given by their sum only if  $k = \hat{k} + 1$ . This is consistent with eqs. (4.30) and (4.31) if and only if  $\hat{k} = 1/2$  and  $k = 3/2$ . The full list of the  $N = 8$  algebra generators in the case of  $\hat{k} = 1/2$ ,  $k = 3/2$  and  $c_8 = 84/11$  is given by

$$\begin{aligned} T^{mn} &= -\frac{i}{2} \left\{ \hat{J}^{mn} + \bar{\psi}\tilde{\gamma}^{mn}\psi \right\} , \\ S^m &= i\sqrt{\frac{7}{33}} \gamma_{ab}^m \psi^a \hat{J}^b , \quad S^8 = \sqrt{\frac{7}{33}} \psi^a \hat{J}^a , \\ T &= \frac{1}{132} \left\{ 7 : \hat{J}^a \hat{J}^a : - : \hat{J}^{mn} \hat{J}^{mn} : - 42 : \bar{\psi} \partial \psi : \right. \\ &\quad \left. + \frac{3}{2} \hat{J}^{mn} : (\bar{\psi}\tilde{\gamma}^{mn}\psi) : - : (\bar{\psi}\tilde{\gamma}^{mn}\psi)(\bar{\psi}\tilde{\gamma}^{mn}\psi) : \right\} . \end{aligned} \tag{4.41}$$

Another consistent solution is to start from the *non-compact* real form  $\widehat{SO(7,1)}$  in our Ansatz, and take its level to be  $\hat{k} = -1/2$ . Using eq. (4.31), this yields the level  $k = 9/2$  for the affine  $SO(7)$  symmetry of the  $N = 8$  algebra, and a central charge  $c_8 = 360/17$  according to eq. (3.5).

In both cases of consistent solutions the corresponding  $(1,0)$  supersymmetric gauged WZNW models with the target space  $SO(8)/SO(7) \times U(1)$  or  $SO(7,1)/SO(7) \times U(1)$  must therefore have a hidden non-linear  $N = 8$  superconformal symmetry on-shell.

(ii). Because of eq. (4.21), we are to distinguish the two possibilities: (a) without a  $U(1)$  current ( $\hat{J}^8 = 0$ ) but with a background charge ( $\gamma \neq 0$ ), and (b) vice versa,  $\hat{J}^8 \neq 0$  but  $\gamma = 0$ .

The analogue of eq. (4.24) in the case (ii),  $\hat{J}^m = 0$  and  $\beta \neq 0$ , is given by

$$\frac{i(k+2)}{6(k+4)} T^{mn} = \alpha^2 \left\{ 2\psi^m \psi^n [2\beta^2 (\bar{B}^2 + \bar{C}^2) - 2i\beta\gamma\bar{B}] - 2C^{mn}_p \psi^p \psi^8 [-2i\beta\gamma\bar{C} + 4\beta^2 \bar{B}\bar{C}] \right\}$$

$$+ C^{mn}{}_{pq} \psi^p \psi^q \left[ -\frac{1}{2} \beta^2 \overline{C}^2 - \frac{1}{16} \hat{k}_1 + i\beta \gamma \overline{B} \right] \} \quad (4.42)$$

$$\frac{i(k+2)}{6(k+4)} C^m{}_{pq} T^{pq} = \alpha^2 \left\{ C^m{}_{pq} \psi^p \psi^q [2i\beta \gamma (\overline{A} - \overline{C}) - 8\beta^2 \overline{A} \overline{B}] + \psi^m \psi^8 [\frac{1}{2} \hat{k}_1 - 12\beta^2 \overline{A} \overline{C}] \right\} .$$

These two equations are only compatible if

$$\begin{aligned} 4\beta^2 \overline{B}^2 + 6\beta^2 \overline{C}^2 + \frac{1}{4} \hat{k}_1 - 4i\beta \gamma \overline{B} &= 2i\beta \gamma (\overline{A} - \overline{C}) - 8\beta^2 \overline{A} \overline{B} , \\ -48i\beta \gamma \overline{C} + 96\beta^2 \overline{B} \overline{C} &= 24\beta^2 \overline{A} \overline{C} - \hat{k}_1 , \end{aligned} \quad (4.43)$$

where we have to add the additional condition  $\gamma \hat{k}_1 = 0$  from eq. (4.21).

It is not difficult to check that there is only one consistent solution of eq. (4.43) which is compatible with the  $SO(7)$  symmetry, namely,

$$\gamma = 0 , \quad \overline{A} = \overline{B} = \overline{C} = 1 , \quad \text{and} \quad \hat{k}_1 = -72\beta^2 . \quad (4.44)$$

The first line of eq. (4.42) thus takes the form

$$T^{mn} = -\frac{i}{2} k (\bar{\psi} \tilde{\gamma}^{mn} \psi) , \quad (4.45)$$

where we have used eq. (4.27) and the eq. (4.7a) gives :

$$\alpha^2 = \frac{k(k+2)}{48\beta^2(k+4)} . \quad (4.46)$$

Eqs. (4.45) and (3.5) now imply that

$$k = 1 , \quad \text{and} \quad c_8 = 26/5 , \quad (4.47)$$

respectively. The list of the  $N = 8$  superconformal algebra generators in the case (ii) is given by

$$\begin{aligned} T^{mn} &= -\frac{i}{2} (\bar{\psi} \tilde{\gamma}^{mn} \psi) , \\ S^m &= \frac{i}{\sqrt{5}} \left( i\psi^m \hat{J}^8 + \frac{1}{3} C^m{}_{npq} \psi^n \psi^p \psi^q + C^m{}_{pq} \psi^p \psi^q \psi^8 \right) , \\ S^8 &= \frac{i}{\sqrt{5}} \left( -i\psi^8 \hat{J}^8 + C_{mnp} \psi^m \psi^n \psi^p \right) , \\ T &= \frac{1}{20} : \hat{J}^8 \hat{J}^8 : - \frac{3}{8} : \bar{\psi} \partial \psi : + \frac{1}{240} : (\bar{\psi} \tilde{\gamma}^{mn} \psi) (\bar{\psi} \tilde{\gamma}^{mn} \psi) : . \end{aligned} \quad (4.48)$$



This case thus corresponds to a unitary realization of the  $N = 8$  algebra in terms of a single free boson and eight free fermions transforming in  $\underline{1}$  and  $\underline{8}$  of  $Spin(7)$ , respectively.

## 4.2 A construction of the exceptional $N = 7$ superconformal algebra over the coset space $SO(7) \times U(1)/G_2$

The  $N = 7$  coset construction over  $SO(7) \times U(1)/G_2$  follows the lines of the  $N = 8$  case considered above. In the  $N = 7$  case, our starting point is the affine algebra  $\widehat{SO(7)}_{\hat{k}}$  whose commutation relations can be read off from eq. (4.12) by restricting the indices to run from one to seven ( $m, n, \dots = 1, \dots, 7$ ). Eqs. (2.17) and (2.20) imply the following definitions of the currents associated with the coset  $SO(7)/G_2$  and the group  $G_2$ , respectively,

$$\hat{A}^m(z) = \frac{1}{2}C^m_{np}\hat{J}^{np}(z) \ , \quad \text{and} \quad \hat{G}^{mn}(z) \equiv \frac{1}{2}\hat{J}^{mn}(z) + \frac{1}{8}C^{mn}_{pq}\hat{J}^{pq}(z) \ . \quad (4.49)$$

Accordingly, we get the OPE

$$\hat{A}^m(z)\hat{A}^n(w) = \frac{-12\hat{k}}{(z-w)^2}\delta^{mn} + \frac{2C^{mn}_k\hat{A}^k - 8\hat{G}^{mn}}{z-w} + : \hat{A}^m\hat{A}^n : (w) + \dots \ . \quad (4.50)$$

We shall denote the affine factor  $\widehat{U(1)}$  by a bosonic current  $\hat{A}^0(z)$ , with the (normalised) OPE (*cf.* eq. (4.13))

$$\hat{A}^0(z)\hat{A}^0(w) = \frac{1/2}{(z-w)^2} + : \hat{A}^0\hat{A}^0 : (w) + \dots \ , \quad (4.51)$$

and define the  $8=1+7$  free fermions  $\psi^a(z)$  to be  $(\psi^m, \psi^8)$ , with the OPE as in eq. (4.4).

Our Ansatz for the supercurrents of the  $N = 7$  non-linear superconformal algebra reads as follows:

$$\begin{aligned} S^m = & \ 2\alpha \left\{ C^m_{np}\psi^n\hat{A}^p + a\psi^8\hat{A}^m + b\psi^m\hat{A}^0 + \gamma\partial\psi^m \right. \\ & \left. - \frac{2i}{3}\beta \left[ C^m_{npq}\psi^n\psi^p\psi^q + 3dC^m_{np}\psi^8\psi^n\psi^p \right] \right\} \ , \end{aligned} \quad (4.52)$$

where  $\alpha, \beta, a, b$  and  $d$  are parameters to be determined by the OPEs of the  $N = 7$  algebra.

In terms of the general Ansatz (4.5), eq. (4.52) implies that we equate

$$h_{np}^m = C_{np}^m, \quad h_{8n}^m = a\delta_n^m, \quad h_{n8}^m = b\delta_n^m; \quad \xi_n^m = \gamma\delta_n^m, \quad (4.53)$$

and

$$\Gamma_{npq}^m = C_{npq}^m, \quad \Gamma_{np8}^m = dC_{np}^m. \quad (4.54)$$

It follows <sup>9</sup>

$$\begin{aligned} h_{ab}^m h_{ab}^n &= (6 + a^2 + b^2) \delta^{mn}, \\ h_{ab}^m h_{ad}^n &= h_{pb}^m h_{pd}^n + h_{8b}^m h_{8d}^n, \end{aligned} \quad (4.55)$$

where

$$\begin{aligned} h_{aq}^m h_{ar}^n &= C_{mnqr} - \delta_{qn}\delta_{mr} + \delta_{mn}\delta_{qr} + a^2\delta_{mq}\delta_{nr}, \\ h_{a8}^m h_{a8}^n &= b^2\delta^{mn}, \\ h_{a8}^m h_{ap}^n &= -bC_{mnp}, \\ h_{ap}^m h_{a8}^n &= +bC_{mnp}, \end{aligned} \quad (4.56)$$

and, similarly,

$$\begin{aligned} h_{pc}^m h_{qc}^n &= C_{mnpq} + \delta_{mn}\delta_{pq} - \delta_{mq}\delta_{pn} + b^2\delta_{mp}\delta_{nq}, \\ h_{8c}^m h_{8c}^n &= a^2\delta^{mn}, \\ h_{8c}^m h_{pc}^n &= +aC_{mnp}, \\ h_{pc}^m h_{8c}^n &= -aC_{mnp}. \end{aligned} \quad (4.57)$$

In addition, we find

$$\Gamma_{abc}^m \Gamma_{abc}^n = 6(4 + 3d^2) \delta^{mn}, \quad (4.58)$$

$$\Gamma_{abc}^m \Gamma_{abd}^n = \Gamma_{pqc}^m \Gamma_{pqd}^n + 2\Gamma_{8pc}^m \Gamma_{8pd}^n,$$

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<sup>9</sup>Note our conventions: the *early* lower-case Latin indices take values  $a, b, \dots = 1, 2, \dots, 8$ , whereas the *middle* lower-case Latin indices take values  $i, j, \dots = 1, 2, \dots, 7$ .

where

$$\begin{aligned}
\Gamma_{abr}^m \Gamma_{abs}^n &= 2(1 + d^2) C^{mn}{}_{rs} + 2(2 + d^2) (\delta_n^m \delta_s^r - \delta_s^m \delta_n^r) , \\
\Gamma_{abr}^m \Gamma_{ab8}^n &= -4d C^{mnr} , \\
\Gamma_{ab8}^m \Gamma_{abr}^n &= +4d C^{mnr} .
\end{aligned} \tag{4.59}$$

Some other useful corollaries of eq. (4.52) are

$$\begin{aligned}
\frac{i}{2} h_{ab}^m h_{ad}^n f^{bdE} \hat{J}^E &= -4(3 + a^2) \hat{G}^{mn} + (a^2 - 3) C^{mnp} \hat{A}^p , \\
h_{ab}^m h_{cb}^n \psi^a \psi^c &= (1 + b^2) \psi^m \psi^n + C^{mn}{}_{pq} \psi^p \psi^q + 2a C^{mnp} \psi^8 \psi^p , \\
2\beta^2 \Gamma_{abc}^m \Gamma_{abd}^n \psi^c \psi^d &= 4\beta^2 (2 + d^2) \psi^m \psi^n + 4\beta^2 (1 + d^2) C^{mn}{}_{pq} \psi^p \psi^q \\
&\quad + 16\beta^2 d C^{mnp} \psi^8 \psi^p .
\end{aligned} \tag{4.60}$$

We are now in a position to calculate the r.h.s. of eqs. (4.8b) and (4.8a), by using the relation  $\tilde{k} = -24\hat{k}$  which follows from eq. (4.50), and the equations above. We find

$$\begin{aligned}
\frac{3k+5}{k+3} (M^A)^{mn} G^A &= 4\alpha^2 \left\{ -4(3 + a^2) \hat{G}^{mn} + (a^2 - 3 + \gamma) C^{mnp} \hat{A}^p \right. \\
&\quad + 4 \left[ \beta^2 (2 + d^2) - 3\hat{k} + \frac{1}{8} b^2 \right] \psi^m \psi^n \\
&\quad + \left[ 4\beta^2 (1 + d^2) - 12\hat{k} - \frac{4i}{3} \beta \gamma \right] C^{mn}{}_{pq} \psi^p \psi^q \\
&\quad \left. + 4 \left[ 4\beta^2 d - 6\hat{k}a + i\gamma\beta d \right] C^{mnp} \psi^8 \psi^p \right\} ,
\end{aligned} \tag{4.61}$$

and

$$\frac{k(3k+5)}{k+3} = 4\alpha^2 \left[ 2\beta^2 (4 + 3d^2) - 6\hat{k}(6 + a^2) + \frac{1}{4} b^2 - \gamma^2 \right] . \tag{4.62}$$

The stress tensor  $T$  of the  $N = 7$  algebra can be calculated from the OPE of the given supercurrents  $S^m$  in eq. (4.52) along the lines of the previous subsection. Taking the trace in eq. (3.11) and using eq. (3.6), we find

$$S^m(z) S^m(w) \sim \frac{7k(3k+5)}{(k+3)(z-w)^3} + \frac{1}{z-w} \left[ 14T(w) + \frac{4}{3(k+3)} : G^{mn} G^{mn} : (w) \right] . \tag{4.63}$$

Since  $\text{tr}(M^A) = 0$  and  $\text{tr}(M^A M^B) = 2\delta^{AB}$ , eqs. (4.8c) and (4.63) imply

$$\begin{aligned}
7T - \frac{3}{\hat{k} + 4} : G^A G^A : &= \alpha^2 \left\{ h_{ab}^m h_{ad}^m : \hat{A}^b \hat{A}^d : - \tilde{k} h_{ab}^m h_{cb}^m : \psi^c \partial \psi^a : \right. \\
&+ 4h_{ap}^m h_{cq}^m (C_s^{pq} \hat{A}^s - 4\hat{G}^{pq}) : \psi^a \psi^c : - 4i\beta \Gamma_{abc}^m h_{cf}^m \hat{A}^f : \psi^a \psi^b : - 4\beta^2 \Gamma_{abf}^m \Gamma_{abg}^m : \psi^g \partial \psi^f : \\
&\left. + 7\gamma b \partial \hat{A}^0 - 4\beta^2 \Gamma_{abc}^m \Gamma_{afg}^m : \psi^b \psi^c \psi^f \psi^g : \right\} , \tag{4.64}
\end{aligned}$$

where  $m, n, \dots = 1, \dots, 7$ , and  $a, b, \dots = 1, \dots, 8$ .

In accordance with our definitions, we have

$$\begin{aligned}
h_{ab}^m h_{ad}^m : \hat{A}^b \hat{A}^d : &= (6 + a^2) : \hat{A}^m \hat{A}^m : + 7b^2 : \hat{A}^0 \hat{A}^0 : , \\
h_{ab}^m h_{cb}^m : \psi^c \partial \psi^a : &= (6 + b^2) : \psi^m \partial \psi^m : + 7a^2 : \psi^8 \partial \psi^8 : , \\
h_{ap}^m h_{cq}^m (C_s^{pq} \hat{A}^s - 4\hat{G}^{pq}) : \psi^a \psi^c : &= 12a \hat{A}^m : \psi^m \psi^8 : - 3(C^{mn}{}_p \hat{A}^p + 4\hat{G}^{mn}) : \psi^m \psi^n : , \\
\Gamma_{abc}^m h_{cf}^m \hat{A}^f : \psi^a \psi^b : &= 12d \hat{A}^m : \psi^m \psi^8 : + (a - 4) C^{mn}{}_p \hat{A}^p : \psi^m \psi^n : , \\
\Gamma_{abf}^m \Gamma_{abg}^m : \psi^g \partial \psi^f : &= 12(2 + d^2) : \psi^m \partial \psi^m : + 42d^2 : \psi^8 \partial \psi^8 : , \tag{4.65}
\end{aligned}$$

and

$$\Gamma_{abc}^m \Gamma_{afg}^m : \psi^b \psi^c \psi^f \psi^g : = -(2 + d^2) C_{mnpq} : \psi^m \psi^n \psi^p \psi^q : - 16d : \psi^m \psi^n \psi^p \psi^8 : . \tag{4.66}$$

Hence, we can rewrite eq. (4.64) as follows:

$$\begin{aligned}
7T - \frac{3}{\hat{k} + 4} : G^A G^A : &= \alpha^2 \left\{ (6 + a^2) : \hat{A}^m \hat{A}^m : + 7b^2 : \hat{A}^0 \hat{A}^0 : + 7\gamma b \partial \hat{A}^0 \right. \\
&+ [144\hat{k} - 48\beta^2(2 + d^2) - b^2] : \psi^m \partial \psi^m : + 168(\hat{k}a^2 - \beta^2 d^2) : \psi^8 \partial \psi^8 : \\
&- 48\hat{G}^{mn} : \psi^m \psi^m : + 4[i\beta(4 - a) - 3] C_{pmn} \hat{A}^p : \psi^m \psi^n : + 48(a - i\beta d) \hat{A}^m : \psi^m \psi^8 : \\
&\left. + 4\beta^2(2 + d^2) C_{mnpq} : \psi^m \psi^n \psi^p \psi^q : + 64\beta^2 d C_{mnp} : \psi^m \psi^n \psi^p \psi^8 : \right\} . \tag{4.67}
\end{aligned}$$

The terms bilinear or quartic in the fermionic fields  $\psi^m$  have to appear in the currents of the  $N = 7$  algebra in a covariant form, i.e. in the  $G_2$ -covariant combination

$$\bar{\psi}g^{mn}\psi \equiv \bar{\psi} \left( \frac{1}{2}\gamma^{mn} + \frac{1}{8}C^{mn}{}_{pq}\gamma^{pq} \right) \psi = 2\psi^m\psi^n + \frac{1}{2}C^{mn}{}_{pq}\psi^p\psi^q . \quad (4.68)$$

Some related identities are given by

$$\begin{aligned} \hat{G}^{mn} : \psi^m\psi^n : &= \frac{1}{3}\hat{G}^{mn} : (\bar{\psi}g^{mn}\psi) , \\ C_{mn}^p \hat{A}^p : \psi^m\psi^n : &= -i\hat{A}^m : (\bar{\psi}\tilde{\gamma}^m\psi) : + 2\hat{A}^m : \psi^m\psi^8 : , \\ C_{mnp} : \psi^m\psi^n\psi^p\psi^8 : &= -i : (\bar{\psi}\tilde{\gamma}^m\psi)\psi^m\psi^8 : , \\ C_{mnpq} : \psi^m\psi^n\psi^p\psi^q : &= \frac{2}{3} : (\bar{\psi}g^{mn}\psi)(\bar{\psi}g^{pq}\psi) : . \end{aligned} \quad (4.69)$$

When using these identities and eqs. (2.11) and (2.19), we arrive at the explicitly  $G_2$ -covariant expression for the stress tensor, namely,

$$\begin{aligned} T = \frac{3}{7(\hat{k} + 4)} : G^A G^A : &+ \frac{\alpha^2}{7} \left\{ (6 + a^2) : \hat{A}^m \hat{A}^m : + 7b^2 : \hat{A}^0 \hat{A}^0 : + 7\gamma b \partial \hat{A}^0 \right. \\ &+ [144\hat{k} - 48\beta^2(2 + d^2) - b^2] : \psi^m \partial \psi^m : + 168(\hat{k}a^2 - \beta^2 d^2) : \psi^8 \partial \psi^8 : \\ &+ 4[\beta(4 - a) + 3i] \hat{A}^m : (\bar{\psi}\tilde{\gamma}^m\psi) : + 8[6(a - i\beta d) + i\beta(4 - a) - 3] \hat{A}^m : \psi^m\psi^8 : \\ &\left. - 16\hat{G}^{mn} : (\bar{\psi}g^{mn}\psi) : - 64i\beta^2 d : (\bar{\psi}\tilde{\gamma}^m\psi)\psi^m\psi^8 : + \frac{8}{3}\beta^2(2 + d^2) : (\bar{\psi}g^{mn}\psi)^2 : \right\} , \end{aligned} \quad (4.70)$$

where  $\hat{A}^m$ ,  $\psi^m$  and  $: (\bar{\psi}\tilde{\gamma}^m\psi) :$  all transform in  $\underline{7}$  of  $G_2$ .

Once all the currents of the  $N = 7$  algebra are determined, we are to consider the consistency conditions which follow from eq. (4.61) and (4.8c). The terms in (4.8c) that are symmetric and traceless with respect to  $(m, n)$  indices determine the composite field  $: GG :$  which should not depend on the coset currents  $\hat{A}^m$ . Similarly to the  $N = 8$  case, it is not difficult to check that this is only possible if either

- (i)  $\hat{A}^m \neq 0$ ,  $\gamma \neq 0$  and  $\beta = d = 0$ ,

or

(ii)  $\hat{A}^m = 0$ ,  $\gamma = 0$  and  $\beta \neq 0$ .

Each case is separately considered below.

(i). It follows from eqs. (4.61) and (4.68) that

$$a^2 + \gamma = 3, \quad a\hat{k} = 0, \quad b^2 + 72\hat{k} = 0. \quad (4.71)$$

Therefore, in order to have a non-trivial solution ( $\hat{k} \neq 0$ ), we must take

$$a = 0, \quad \gamma = 3, \quad b^2 = -72\hat{k}. \quad (4.72)$$

Eqs. (4.61) and (4.62) now imply (*cf.* eq. (4.30))

$$G^A = -\frac{1}{3} (M^A)^{mn} \left[ \frac{2k}{1+6\hat{k}} \right] \left\{ \hat{G}^{mn} + 2\hat{k} (\bar{\psi} g^{mn} \psi) \right\}, \quad (4.73)$$

and

$$\frac{k(3k+5)}{k+3} = -36\alpha^2(1+6\hat{k}). \quad (4.74)$$

Since the relative normalisation of the  $G_2$  generators  $\hat{G}^{mn}$  and  $(\bar{\psi} g^{mn} \psi)$  is the same, it follows from eq. (4.73) that  $2\hat{k} = 1$ , which implies  $k = \hat{k} + 1 = 3/2$ , exactly as in the  $N = 8$  case ! This is also consistent with the level equation associated with eq. (4.73).

The affine currents of the  $N = 7$  algebra now take the very simple form:

$$G^A(z) = -\frac{1}{4} (M^A)^{mn} \left\{ \hat{G}^{mn} + (\bar{\psi} g^{mn} \psi) \right\}, \quad (4.75)$$

which is quite similar to the expression for the  $N = 8$  affine currents in eq. (4.41).

Eqs. (3.6) and (4.75) also imply

$$\begin{aligned} : G^A G^A : &= -\frac{1}{8} : (\hat{G}^{mn} + \bar{\psi} g^{mn} \psi)^2 : \\ &= -\frac{1}{8} : [ : \hat{G}^{mn} \hat{G}^{mn} : + 2\hat{G}^{mn} : (\bar{\psi} g^{mn} \psi) : + : (\bar{\psi} g^{mn} \psi)^2 : ] . \end{aligned} \quad (4.76)$$

The remaining coefficient values are therefore given by  $\alpha = (i/12)\sqrt{19/6}$  and  $b = i6$ . Eq. (3.12) for  $k = 3/2$  yields  $c_7 = 89/12$ . The list of the  $N = 7$  currents in this realization reads as follows:

$$\begin{aligned}
G^A &= -\frac{1}{4}(M^A)^{mn} \left\{ \hat{G}^{mn} + \bar{\psi} g^{mn} \psi \right\} , \\
S^m &= \frac{i}{6} \sqrt{\frac{19}{6}} \left\{ C^m_{np} \psi^n \hat{A}^p + i6 \psi^m \hat{A}^0 + 3 \partial \psi^m \right\} , \\
T &= -\frac{1}{84} : \left( \hat{G}^{mn} + \bar{\psi} g^{mn} \psi \right)^2 : - \frac{19}{7 \cdot 12^2} \left\{ : \hat{A}^m \hat{A}^m : - 42 : \hat{A}^0 \hat{A}^0 : + i21 \partial \hat{A}^0 \right. \\
&\quad \left. + 18 : \psi^m \partial \psi^m : - 16 \hat{G}^{mn} : (\bar{\psi} g^{mn} \psi) : + 12 i \hat{A}^m : (\bar{\psi} \tilde{\gamma}^m \psi) : \right\} ,
\end{aligned} \tag{4.77}$$

where  $\psi^8 = 0$ . It should be noted that the currents  $\hat{A}^m$  are anti-hermitian. This leads to the extra factors of  $i$  in the expressions for  $S^m$  and  $T$  so as to make all the terms appearing in them hermitian, as required by unitarity.

(ii). Another realization of the  $N = 7$  algebra is possible in terms of one real boson and seven real fermions at the level  $k = 1$ . It corresponds to the following solution of the consistency equations:

$$a = d = \gamma = 0 , \quad b = 4\beta , \quad \alpha^2 \beta^2 = 1/24 . \tag{4.78}$$

The list of the  $N = 7$  currents in this case of  $c_7 = 5$  reads:

$$\begin{aligned}
G^A &= -\frac{1}{3}(M^A)^{mn} (\bar{\psi} g^{mn} \psi) , \\
S^m &= \frac{2}{\sqrt{6}} \left\{ 2 \psi^m \hat{A}^0 - \frac{i}{3} C^m_{npq} \psi^n \psi^p \psi^q \right\} , \\
T &= \frac{2}{3} : \hat{A}^0 \hat{A}^0 : - \frac{2}{3} : \psi^m \partial \psi^m : + \frac{1}{126} : (\bar{\psi} g^{mn} \psi) (\bar{\psi} g^{mn} \psi) : .
\end{aligned} \tag{4.79}$$

It is straightforward to check the rest of the  $N = 7$  algebra.

## 5 Conclusion

The main results of our investigation are given in sect. 4 where we presented explicit realizations of the  $N = 7$  and  $N = 8$  non-linear superconformal algebras using coset space methods. The constraints of the non-linear algebras allowed very few realizations for each algebra in the compact case, as well as an additional realization for the  $N = 8$  algebra in the non-compact case, within our general Ansatz over specific coset spaces. It may be possible to find other realizations by considering other more general coset spaces.

Our results could be relevant for string theory in various ways. The exceptional superconformal symmetries may arise as hidden symmetries in certain compactifications of superstring theories. For example, there exist octonionic soliton solutions to the low-energy heterotic string theory [47, 43, 48]. The octonionic soliton of ref. [47] is related to the Yang-Mills instanton in eight dimensions with  $SO(7)$  gauge group [49, 50]. The octonionic soliton of ref. [43] is related to the seven-dimensional Yang-Mills instanton with the gauge group  $G_2$ , which is a remarkable instanton in odd dimensions [43] (see also ref. [51] for other examples). The conformal field-theoretic formulations of these remarkable octonionic solitons of the heterotic string may involve the exceptional superconformal algebras.

As was shown in ref. [52], the light-cone gauge actions of various superstring theories in the Green-Schwarz formalism have  $N = 8$  supersymmetry. Since they are also conformally invariant, this implies that they must be invariant under some  $N = 8$  supersymmetric extension of the Virasoro algebra. Later, the Green-Schwarz superstring was shown to have the  $N = 8$  soft algebra [12, 13, 14, 47] as part of its constraint algebra. Whether the Green-Schwarz superstring action has a hidden non-linear  $N = 8$  or  $N = 7$



supersymmetry of the type investigated above is an interesting open problem.

A compactification of eleven-dimensional supergravity on the 7-sphere  $S^7$  is known to give  $N = 8$  supergravity in four dimensions, with the  $SO(8)$  gauge invariance [53, 54]. Similarly, the cosets  $S^7 \times S^1$  or  $S^7 \times S^1 \times S^1$  could be used to compactify the eleven-dimensional supergravity down to three or two dimensions. Recently, it has been realized that an  $S^1$ -compactified eleven-dimensional supergravity appears to be the low-energy effective field theory of the strongly coupled type-IIA superstring which, in turn, is related to the  $S^1$ -compactified eleven-dimensional supermembrane [55, 56]. In general, massless non-Abelian gauge fields do not arise as Kaluza-Klein modes in consistent string compactifications. It is however possible that the 11-dimensional M-theory may have consistent compactification over manifolds with non-trivial isometry groups such as the seven-sphere. If that is the case, then the  $N = 8$  and  $N = 7$  superconformal algebras may be the hidden symmetries of compactifications involving the seven-sphere.

Interestingly enough, there exist Ricci-flat seven- and eight-dimensional compact manifolds with the holonomy groups  $G_2$  and  $Spin(7)$ , respectively [57]. Both the ten-dimensional superstrings and the eleven-dimensional supergravity compactified down to two and three dimensions ( and three and four dimensions) on such manifolds have the minimal number (one) of supersymmetries in the corresponding dimension [58, 59]. Whether or not the exceptional superconformal algebras can be related to these constructions remains to be investigated.

Having obtained the unitary realizations of the exceptional superconformal algebras with the central charges  $c = 26/5$  and  $c = 5$ , one can use them to represent the  $N = 0$  and  $N = 1$  superconformal matter. Then, by ‘tensoring’ five conformal matter models of  $c = 26/5$  and adding the conformal ghosts  $(b, c)$ , one gets an anomaly-free string model ( $c_{\text{gh}} =$

$-26$ ). Similarly, by ‘tensoring’ three superconformal matter models of  $c = 5$  and adding both the conformal and superconformal ghosts  $(b, c)$  and  $(\beta, \gamma)$ , one gets an anomaly-free superstring model ( $c_{\text{gh}} = -26 + 11 = -15$ ). Thus the exceptional superconformal algebras may describe ‘exceptional’ compactifications of existing superstring theories and underlie some novel ‘exceptional’ strings and superstrings !

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## Appendix A: Identities for the octonionic structure constants $C^{mnp}$ and $C^{mnpq}$

The identities collected below follow from the definitions of tensors  $C^{mnp}$  and  $C^{mnpq}$  in subsect. 2.1. We find (*cf* refs. [41, 40])

$$\begin{aligned} C^p{}_{mn} C^{mn}{}_q &= 6 \delta_q^p, & C^{mnp} C^{mnp} &= 42, \\ C^{[q}{}_{[mn} C^{ts]}{}_{p]} &= -2 C^{[qt}{}_{[mn} \delta_p^s], \end{aligned} \tag{A.1}$$

$$\begin{aligned} C^p{}_{mk} C^q{}_{kn} &= \frac{1}{2} C^{pq}{}_k C_{knm} - \frac{3}{2} C^{pq}{}_{mn} + \frac{1}{2} (\delta_m^p \delta_n^q + \delta_m^q \delta_n^p) - \delta^{pq} \delta_{mn}; \\ C_{mnkl} C^{kl}{}_p &= -4 C_{mnp}, \\ 2C^k{}_{[mn} C^{ks}{}_{pq]} &= C^k{}_{[mn} C^{ks}{}_{p]q} - C^k{}_{q[m} C^{ks}{}_{np]}, \\ C^{k[s}{}_{[m} C^{t]k}{}_{np]} &= C^{[t}{}_{[mn} \delta_p^s], \\ C^{ks}{}_{[m} C^{tk}{}_{np]} &= C_{mnp} \delta^{st} - C^s{}_{[mn} \delta_p^t - 2C^t{}_{[mn} \delta_p^s], \\ C^k{}_{[mn} C^{pq}{}_{s]k} &= 4 C^{[p}{}_{[mn} \delta_s^q], \\ C_{mnk} C^{kpqs} &= 6 C^{[pq}{}_{[m} \delta_n^s], \\ C^{tk}{}_{[mn} C^k{}_{pq]} &= -2 C_{[mnp} \delta_q^t], \\ C_{[mnp} \delta_q^s - C_{[mnp} \delta_s^q] &= \frac{3}{2} C^{[s}{}_{[mn} \delta_p^q]; \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} C_{mnpk} C^{kqst} &= 9 C^{[qs}{}_{[mn} \delta_p^t] + 6 \delta_{[m}^{[q} \delta_n^s \delta_p^t], \\ C_{mnpq} C^{pq}{}_{st} &= -2 C_{mnst} + 4 (\delta_{ms} \delta_{nt} - \delta_{mt} \delta_{ns}), \\ C_{mkpq} C_{nkpq} &= 24 \delta_{mn}, \\ C^{sk}{}_{[mn} C^{tk}{}_{pq]} &= -\delta^{st} C_{mnpq} - C^s{}_{[mn} C^t{}_{pq]} + 2C_{[mnp}{}^s \delta_q^t + 2C_{[mnp}{}^t \delta_q^s]. \end{aligned} \tag{A.3}$$

## Appendix B: $N=8$ supersymmetry algebra

In this appendix, the  $N=8$  supersymmetry algebra of eq. (3.4) is written down in a more explicit form, after using the decomposition  $\underline{8} = \underline{1} + \underline{7}$  for the  $SO(7)$  spinors  $S^a = (S^m, S^8)$ , and substituting the gamma matrices in terms of the octonionic structure constants, as in sect. 2. We find

$$\begin{aligned}
S^8(z)S^8(w) &\sim \frac{8k(k+2)}{3(k+4)} \frac{1}{(z-w)^3} + \frac{2T(w)}{z-w} \\
&\quad - \frac{1}{3(k+4)} \frac{1}{(z-w)} \left\{ : T^{pq}T^{pq} : + \frac{1}{4}C_{mnpq} : T^{mn}T^{pq} : \right\} (w) \\
&= \frac{8k(k+2)}{3(k+4)} \frac{1}{(z-w)^3} + \frac{2T(w)}{z-w} \\
&\quad + \frac{1}{(k+4)} \frac{1}{(z-w)} \left\{ \frac{1}{3} : A^p A^p : - \frac{1}{2} : T^{pq}T^{pq} : \right\} (w) ,
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
S^m(z)S^m(w) &\stackrel{\text{no sum over } m}{\sim} \frac{8k(k+2)}{3(k+4)} \frac{1}{(z-w)^3} + \frac{2T(w)}{z-w} - \frac{1}{3(k+4)} \frac{1}{(z-w)} \\
&\quad \times \left\{ : T^{pq}T^{pq} : + \frac{1}{4}C_{rspq} : T^{rs}T^{pq} : + 2C_{mnpq} : T^{pq}T^{nm} : \right\} (w) \\
&= \frac{8k(k+2)}{3(k+4)} \frac{1}{(z-w)^3} + \frac{2T(w)}{z-w} \\
&\quad + \frac{1}{(k+4)} \frac{1}{(z-w)} \left\{ \frac{1}{3} : A^p A^p : - \frac{1}{2} : T^{pq}T^{pq} : \right\} (w) \\
&\quad + \frac{8}{9(k+4)} \frac{1}{(z-w)} \left\{ C^{mpq} : A_q T^{pm} : - 2 : G^{mq}T^{qm} : \right\} (w) ,
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
S^m(z)S^8(w) &\sim \frac{i2(k+2)}{3(k+4)} C_{pq}^m \left[ \frac{T^{pq}(w)}{(z-w)^2} + \frac{1}{2} \frac{\partial T^{pq}(w)}{z-w} \right] \\
&\quad + \frac{1}{3(k+4)} C_{npq} \frac{: T^{pq}T^{nm} : (w)}{(z-w)} \\
&= \frac{i4(k+2)}{3(k+4)} \left[ \frac{A^m(w)}{(z-w)^2} + \frac{1}{2} \frac{\partial A^m(w)}{(z-w)} \right] \\
&\quad + \frac{2}{3(k+4)} \frac{: A_p T^{pm} : (w)}{z-w} ,
\end{aligned} \tag{B.3}$$

and

$$\begin{aligned}
S^m(z)S^n(w) &\stackrel{m \neq n}{\sim} \frac{i2(k+2)}{3(k+4)} \left[ C^{mn}_{pq} + 2\delta_p^m \delta_q^n \right] \left( \frac{T^{pq}(w)}{(z-w)^2} + \frac{1}{2} \frac{\partial T^{pq}(w)}{z-w} \right) \\
&\quad - \frac{1}{3(k+4)} \frac{1}{(z-w)} [C^{ms}_{pq} : T^{pq} T^{sn} : + C^{ns}_{pq} : T^{pq} T^{sm} :] (w) \\
&= \frac{i4(k+2)}{9(k+4)} \left\{ \frac{8G^{mn}(w) - C^{mn}_p A^p(w)}{(z-w)^2} + \frac{4\partial G^{mn}(w) - \frac{1}{2} C^{mn}_p \partial A^p(w)}{z-w} \right\} \\
&\quad - \frac{4}{9(k+4)} \frac{1}{(z-w)} [2 : G^{ms} T^{sn} : (w) + 2 : G^{ns} T^{sm} : (w) \\
&\quad - C^{ms}_p : A^p T^{sn} : (w) - C^{ns}_p : A^p T^{sm} : (w)] , \\
\end{aligned} \tag{B.4}$$

where we have extensively used the identities from Appendix A. Note also that in this appendix the quantity  $A^m$  represents  $\frac{1}{2} C^m_{np} T^{np}$ .

## References

- [1] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, Cambridge: Cambridge Univ. Press, 1987.
- [2] J. Fuchs, *Affine Lie Algebras and Quantum Groups*, Cambridge: Cambridge Univ. Press, 1992.
- [3] S. V. Ketov, *Conformal Field Theory*, Singapore: World Scientific, 1995.
- [4] M. Günaydin, G. Sierra and P. K. Townsend, Nucl. Phys. **B274** (1986) 429.
- [5] P. Ramond and J. H. Schwarz, Phys. Lett. **64B** (1976) 75.
- [6] L. Alvarez-Gaumé and D. Z. Freedman, Phys. Lett. **94B** (1980) 171;  
Phys. Rev. **D22** (1980) 846; Commun. Math. Phys. **80** (1981) 443.
- [7] P. Spindel, A. Sevrin, W. Troost and A. van Proeyen, Nucl. Phys. **B308** (1988) 662, *ibid.* **311** (1988) 465.
- [8] Z. Hasciewicz, K. Thielemans and W. Troost, J. Math. Phys. **31** (1990) 744.
- [9] V. G. Knizhnik, Theor. Math. Phys. **66** (1986) 68.
- [10] M. Bershadsky, Phys. Lett. **174B** (1986) 285.
- [11] F. Englert, A. Sevrin, W. Troost, A. van Proeyen and P. Spindel, J. Math. Phys. **29** (1988) 281.
- [12] N. Berkovits, Nucl. Phys. **B358** (1991) 169.
- [13] L. Brink, M. Cederwall and C. Preitschopf, Phys. Lett. **311B** (1993) 76.

- [14] M. Cederwall and C. R. Preitschopf, Commun. Math. Phys. **167** (1995) 373.
- [15] J. A. H. Samtleben, Nucl. Phys. **B453** (1995) 429.
- [16] H. Nicolai and N. P. Warner, Commun. Math. Phys. **125** (1989) 369.
- [17] E. Bergshoeff, E. Sezgin and H. Nishino, Phys. Lett. **186B** (1987) 167.
- [18] B. de Wit, A. K. Tollstén and H. Nicolai, Nucl. Phys. **B392** (1993) 3.
- [19] E. S. Fradkin and V. Yu. Linetsky, Phys. Lett. **275B** (1992) 345; *ibid.* **282** (1992) 352.
- [20] P. Bowcock, Nucl. Phys. **B381** (1992) 415.
- [21] K. Ito, J. O. Madsen and J. L. Petersen, Phys. Lett. **318B** (1993) 315;  
Nucl. Phys. **B398** (1993) 425.
- [22] V. G. Kač, *Infinite Dimensional Lie Algebras. An Introduction*, Boston: Birkhäuser, 1983.
- [23] P. G. O. Freund and I. Kaplansky, J. Math. Phys. **17** (1976) 228.
- [24] M. Scheunert, W. Nahm and V. Rittenberg, J. Math. Phys. **17** (1976) 1626.
- [25] B. de Witt and P. van Nieuwenhuizen, J. Math. Phys. **23** 1953.
- [26] M. Günaydin, J. Math. Phys. **31** (1990) 1776;  
*The Exceptional Superspace and the Quadratic Jordan Formulation of Quantum Mechanics*, in “Elementary Particles and the Universe: Essays in Honor of Murray Gell-Mann” , ed. by J. H. Schwarz, Cambridge: Cambridge Univ. Press, 1991.
- [27] P. Goddard and A. Schwimmer, Phys. Lett. **214B** (1988) 209.

- [28] M. Günaydin, J. L. Petersen, A. Taormina and A. van Proeyen, Nucl. Phys. **B322** (1989) 402.
- [29] K. Schoutens, Nucl. Phys. **B292** (1987) 150; *ibid.* **295** (1988) 634; *ibid.* **314** (1989) 519.
- [30] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Commun. Math. Phys. **124** (1989) 209.
- [31] Z. Khviengia and E. Sezgin, Phys. Lett. **326B** (1994) 243.
- [32] S. V. Ketov, Class. and Quantum Grav. **12** (1995) 925; Mod. Phys. Lett. **A10** (1995) 79.
- [33] P. Goddard, A. Kent and D. Olive, Commun. Math. Phys. **103** (1986) 105.
- [34] Y. Kazama and H. Suzuki, Phys. Lett. **216B** (1989) 112; Nucl. Phys. **B321** (1989) 232.
- [35] A. van Proeyen, Class. and Quantum Grav. **6** (1989) 1501.
- [36] A. Sevrin and G. Theodoridis, Nucl. Phys. **B332** (1990) 380.
- [37] M. Günaydin and S. Hyun, Nucl. Phys. **B373** (1992) 688;  
M. Günaydin, Phys. Rev. **D47** (1993) 3600.
- [38] S. J. Gates Jr., and S. V. Ketov, Phys. Rev. **D52** (1995) 2278.
- [39] M. Günaydin and F. Gürsey, J. Math. Phys. **14** (1973) 1651;
- [40] E. Corrigan, C. Devchand, D. B. Fairlie and J. Nuyts, Nucl. Phys. **B214** (1983) 452.



- [41] B. de Wit and H. Nicolai, Nucl. Phys. **B231** (1984) 506.
- [42] R. Dündarar, F. Gürsey and C. Tze, Nucl. Phys. **B266** (1986) 440.
- [43] M. Günaydin and H. Nicolai, Phys. Lett. **351B** (1995) 169.
- [44] J. Wess and B. Zumino, Phys. Lett. **37B** (1971) 95;  
S. Novikov, Sov. Math. Dokl. **24** (1981) 222, Usp. Mat. Nauk **37** (1982) 3;  
E. Witten, Commun. Math. Phys. **92** (1984) 455.
- [45] P. Di Vecchia, V. G. Knizhnik, J. L. Petersen and P. Rossi, Nucl. Phys. **B253**  
(1985) 701.
- [46] P. Mathieu, Phys. Lett. **218B** (1989) 185.
- [47] J. Harvey and A. Strominger, Phys. Rev. Lett. **66** (1991) 549.
- [48] T. A. Ivanova, Phys. Lett. **315B** (1993) 277.
- [49] D. B. Fairlie and J. Nuyts, J. Phys. **A17** (1984) 431.
- [50] S. Fubini and H. Nicolai, Phys. Lett. **155B** (1985) 369.
- [51] T. A. Ivanova and A. D. Popov, Lett. Math. Phys. **24** (1992) 85.
- [52] M. Günaydin, B. E. W. Nilsson, G. Sierra and P. K. Townsend Phys. Lett. **176B**  
(1986) 45.
- [53] M. J. Duff, B. E. W. Nilsson and C. N. Pope, Phys. Rep. **130** (1986) 1.
- [54] B. de Wit and H. Nicolai, Nucl. Phys. **B281** (1987) 211.
- [55] P. K. Townsend, Phys. Lett. **350B** (1995) 184.

- [56] E. Witten, Nucl. Phys. **B443** (1995) 85; *Some comments on string dynamics*, Princeton preprint IASSNS-HEP-95-63, July 1995, hep-th/9507121.
- [57] D. D. Joyce, *Compact Riemannian 7-manifolds with holonomy  $G_2$* , Oxford preprints I and II, 1994; *Compact Riemannian 8-manifolds with exceptional holonomy  $Spin(7)$* , Oxford preprint, 1994.
- [58] S. Shatashvili and C. Vafa, *Superstrings and manifolds with exceptional holonomy*, Harvard and Princeton preprint, HUTP-94/A016 and IASSNS-HEP-94/47, 1994, hep-th/9407025.
- [59] G. Papadopoulos and P. K. Townsend, Phys. Lett. **357B** (1995) 300.